MATH45061: EXAMPLE SHEET[1] V

1.) A unit cube of homogeneous, isotropic hyperelastic material undergoes a deformation such that the lengths of its sides become λ_1, λ_2 and λ_3, but its faces remain aligned with the same coordinate planes. You may assume that λ_i is constant.

a.) Explain why the deformed position can be given by

$$X = \lambda_1 x, \quad Y = \lambda_2 y, \quad Z = \lambda_3 z,$$

if (x, y, z) are Cartesian coordinates aligned with the original cube.

b.) Find the strain invariants in terms of the stretches λ_i.

c.) Use the general hyperelastic constitutive law to find the components of the Cauchy (body) stress T^{ij} and explain why the body is in equilibrium.

d.) If the body is incompressible and loaded by a tension parallel to the x-axis. Find the physical stress component $\sigma_{(11)}$, assuming that the stresses within the $y - z$ planes are zero and that the in-plane stretch is the same in all directions. What happens if $\lambda_2 = \lambda_3 = 1$?

e.) What is the physical stress component if the stretch is infinitesimal, $\lambda_1 = 1 + \epsilon\tilde{\lambda}$?

2.) Show that under infinitesimal deformation, the three strain invariants are approximated to $\mathcal{O}(\epsilon)$ by

$$I_1 = 3 + 2e^k_k, \quad I_2 = 3 + 4e^k_k, \quad I_3 = 1 + 2e^k_k.$$

For I_3, work in Cartesians!

3.) By using the equation $s^{ij} = \frac{\partial \mathcal{W}}{\partial \gamma_{ij}}$, show that when the deformations are small the St. Venant–Kirchhoff strain energy function is consistent with the constitutive law for a linear, isotropic, homogeneous material in the absence of pre-stress or thermal effects.

4.) A cylinder of incompressible, hyperelastic material with undeformed radius a and undeformed length l is rotating steadily about its axis with angular velocity ω. We define a cylindrical polar coordinate system (r, θ, z) aligned in the deformed body and work in a rotating frame so that the rotation is represented by a body force per unit mass $r\omega^2\hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is a unit vector directed away from the axis of the cylinder in a plane perpendicular to the axis.

a.) Assuming that there is a constant uniform stretch μ along the axis of the cylinder, explain why the undeformed position in global Cartesian coordinates is given by

$$x_1 = r\sqrt{\mu}\cos\theta, \quad x_2 = r\sqrt{\mu}\sin\theta, \quad x_3 = z/\mu.$$

Hint: Remember the incompressibility constraint.

[1]Any feedback to: *Andrew.Hazel@manchester.ac.uk*

b.) Find the stress component T^{11} in the cylinder, assuming that the curved surface is traction free. Does the result depend on the strain energy function? Hint: You will need to solve the equilibrium equation to find the unknown pressure field P, for which it useful to know that $G^{ij}||_i = 0$. (Prove this by transforming to Cartesian coordinates).

5.) Consider a deformable body that is subject to mixed stress and displacement boundary conditions such that

$$T^{ij}N_j = t^i, \quad \text{on} \quad \partial\Omega_t^s,$$
$$\boldsymbol{R} = \boldsymbol{X}, \quad \text{on} \quad \partial\Omega_t^r.$$

Let $\boldsymbol{R}^*$ and $\widehat{\boldsymbol{R}}$ be two deformations that are consistent with displacement boundary conditions and $\widehat{\mathsf{T}}$ be a stress corresponding to $\widehat{\boldsymbol{R}}$ in the sense that the body force $\widehat{\boldsymbol{F}}$ and surface traction $\widehat{\boldsymbol{t}}$ are chosen so that the body is in equilibrium

$$\widehat{T}^{ij}||_j + \rho\widehat{F}^i = 0, \quad \text{and} \quad \widehat{T}^{ij}N_j = \hat{t}^i, \quad \text{on} \quad \partial\Omega_t^s.$$

a.) Show that

$$\int_{\partial\Omega_t} \widehat{T}^{ij}N_iR_j^* \,\mathrm{d}\mathcal{S}_t = \int_{\partial\Omega_t^r} \widehat{T}^{ij}N_iX_j \,\mathrm{d}\mathcal{S}_t + \int_{\partial\Omega_t^s} \hat{t}^j R_j^* \,\mathrm{d}\mathcal{S}_t,$$
$$= \int_{\Omega_t} \widehat{T}^{ij}R_{j,i}^* \,\mathrm{d}\mathcal{V}_t - \int_{\Omega_t} \rho\widehat{F}^j R_j^* \,\mathrm{d}\mathcal{V}_t.$$

b.) By letting $\widehat{\boldsymbol{R}} = \boldsymbol{R}$ and $\widehat{\mathsf{T}} = \mathsf{T}$ correspond to an equilibrium configuration and $\delta\boldsymbol{R} = \boldsymbol{R}^* - \boldsymbol{R}$, deduce the principle of virtual work for this situation.

c.) Now suppose that we apply a virtual stress $\delta\mathsf{T} = \mathsf{T}^* - \mathsf{T}$, where the two stress states satisfy the traction boundary conditions, but we keep the deformation fixed at $\boldsymbol{R}$, find the associated virtual stress principle.

6.) A state of plane strain is one in deformation remains entirely planar, aligned with the plane spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the global Cartesian coordinate system. The undeformed position is therefore given by

$$\boldsymbol{r}(\xi^1, \xi^2, \xi^3) = \boldsymbol{m}(\xi^1, \xi^2) + \xi^3\boldsymbol{g}_3,$$

where $\boldsymbol{g}_3$ is a unit vector normal to the plane.

a.) Explain why the deformed position is given by

$$\boldsymbol{R}(\xi^1, \xi^2, \xi^3) = \boldsymbol{M}(\xi^1, \xi^2) + \xi^3\boldsymbol{g}_3.$$

b.) Show that under plane strain the stress tensor is of the form

$$T^{ij}(\xi^1, \xi^2) = \begin{pmatrix} T^{11} & T^{12} & 0 \\ T^{12} & T^{22} & 0 \\ 0 & 0 & T^{33} \end{pmatrix}.$$

c.) Show that if the body force is conservative and acts only in the plane, *i. e.* $\boldsymbol{F} = -U_{,\alpha}\boldsymbol{G}^{\alpha}$, the equation of equilibrium reduces to the equation

$$\left[\boldsymbol{T}^{\alpha} - \rho\sqrt{G}U\boldsymbol{G}^{\alpha}\right]_{,\alpha} = \boldsymbol{0}, \tag{1}$$

in the plane, where $\alpha = 1, 2$ and $\boldsymbol{T}^{\alpha} = \sqrt{G}T^{\alpha\beta}\boldsymbol{G}_{\beta}$.

d.) Explain why the expression

$$\boldsymbol{T}^{\alpha} = \sqrt{G}\epsilon^{\gamma\alpha}\boldsymbol{\chi}_{,\gamma} + \rho\sqrt{G}U\boldsymbol{G}^{\alpha},$$

where $\boldsymbol{\chi} = \chi^{\beta}\boldsymbol{G}_{\beta}$ is a vector in the plane $\xi^3 = 0$ and $\epsilon^{\gamma\alpha}$ is the two dimensional Levi-Civita symbol is a solution to equation (1). (**Hint:** Use the symmetry properties of $\sqrt{G}\epsilon^{\gamma\alpha}$, which takes the values: 1 when $\gamma = 1$, $\alpha = 2$; -1 when $\gamma = 2$, $\alpha = 1$; and 0 otherwise.)

e.) Hence show that

$$T^{\alpha\beta} = \epsilon^{\gamma\alpha}\chi^{\beta}|_{\gamma} + \rho U G^{\alpha\beta},$$

and use the symmetry properties of $T^{\alpha\beta}$ and $G^{\alpha\beta}$ to show that

$$\epsilon^{\gamma\alpha}\chi^{\beta}|_{\gamma} = \epsilon^{\gamma\beta}\chi^{\alpha}|_{\gamma},$$

and consequently that

$$\chi^{\beta} = \epsilon^{\delta\beta}\phi_{,\delta}.$$

The scalar function ϕ is called the Airy stress function.

f.) Finally, show that the stress can be written in the form

$$T^{\alpha\beta} = \epsilon^{\gamma\alpha}\epsilon^{\delta\beta}\phi|_{\delta\gamma} + \rho U G^{\alpha\beta},$$

and show that in Cartesian coordinates, in the absence of body forces

$$T_{11} = \phi_{,22}, \quad T_{22} = \phi_{,11}, \quad T_{12} = -\phi_{,12}.$$

7.) A linear, homogeneous, isotropic thermoelastic body in the absence of body forces and heating is governed by the coupled set of equations, after scaling so that the time-derivative terms have no pre-multipliers,

$$\begin{aligned}\frac{\partial^2\boldsymbol{u}}{\partial t^2} &= (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \mu\boldsymbol{\nabla}^2\boldsymbol{u} + \alpha\boldsymbol{\nabla}\theta,\\ \frac{\partial\theta}{\partial t} &= \kappa\boldsymbol{\nabla}^2\theta + \nu\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{u}}{\partial t}.\end{aligned}$$

a.) Show that plane wave solutions of the form

$$\boldsymbol{u} = \boldsymbol{U}\mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)},$$

$$\theta = \Theta\mathrm{e}^{\mathrm{i}(n\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)},$$

can exist if

$$\omega^2\boldsymbol{U} = n^2\left[\mu\boldsymbol{U} + (\lambda + \mu)(\boldsymbol{U}\cdot\boldsymbol{k})\boldsymbol{k}\right] - \mathrm{i}\alpha n\Theta\boldsymbol{k},$$

and

$$(\kappa n^2 - \mathrm{i}\omega)\Theta = \nu n\omega\boldsymbol{U}\cdot\boldsymbol{k}.$$

b.) Show that non-trivial transverse (shear) waves ($\boldsymbol{U} \cdot \boldsymbol{k} = 0$) are independent of thermal effects.

c.) Show that non-trivial longitudinal waves ($\boldsymbol{U} \cdot \boldsymbol{k} = |\boldsymbol{U}|$) are not independent of thermal effects.

8.) A particular strain energy function is given by

$$\mathcal{W} = \frac{C_1}{2}(I_1 - 3) + \frac{C_2}{2}(I_2 - 3) + \frac{c}{2}(I_3 - 1) + \frac{d}{2}(I_3 - 1)^2,$$

where C_1, C_2, c and d are constants.

a.) Show that the corresponding second Piola–Kirchhoff stress tensor is

$$s^{ij} = C_1 g^{ij} + C_2 B^{ij} + (c + 2d(I_3 - 1))I_3 G^{ij}.$$

By using the results from question 2, show that in the infinitesimal limit

$$\tau^{ij} = (C_1 + 2C_2 + c)g^{ij} + (2C_2 + 4d + 2c)e^k_k g^{ij} + (-2C_2 - 2c)e^{ij}.$$

Hint: You will need to establish the result that

$$G^{ij} = [G_{ij}]^{-1} \approx g^{ij} - 2e^{ij}.$$

b.) Hence find the values of C_1, C_2, c and d to ensure that the strain energy function is consistent with the linear, isotropic constitutive law in the absence of heating and pre-stress.