MATH45061: EXAMPLE SHEET[1] III

1.) A body is loaded by a body force $\boldsymbol{F}$ per unit mass and a surface traction $\boldsymbol{T}$. Show that if the linear momentum and the angular momentum about a particular point $\boldsymbol{Z}$ are both in balance, then the angular momentum about any point is also in balance. (You do not need to assume that the body is in equilibrium). Is the result still true if non-zero body couples are present?

2.) Use Cauchy's pillbox argument to prove that

$$\boldsymbol{T}(\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R}) = -\boldsymbol{T}(-\boldsymbol{N}(\boldsymbol{R}), \boldsymbol{R}).$$

3.) By considering the limit of the infinitesimal tetrahedron that has three edges aligned in the general coordinate directions $\chi^{\bar{i}}$, prove the assertion in the lecture notes that

$$\boldsymbol{N}\,\mathrm{d}S = \sum_{\bar{i}} \frac{G^{\bar{i}}\,\mathrm{d}S_{(\bar{i})}}{\sqrt{G^{\bar{i}\bar{i}}}}.$$

4.) In Cartesian coordinates, X_I the force per unit deformed area of a surface with outer unit normal $\boldsymbol{N}$ is given by the stress vector, $\boldsymbol{T}$ with components $T_I = T_{JI}N_J$, where T_{IJ} are the Cartesian components of the Cauchy stress tensor. By considering a change of coordinates to a new Cartesian system $\widehat{\boldsymbol{X}}$ where

$$X_{\widehat{I}} = Q_{\widehat{I}J}X_J,$$

and Q is an orthogonal matrix, show that the Cauchy stress does indeed satisfy tensor transformation properties.

5.) A solid body is immersed in a liquid of constant density ρ_f. The surface traction on the body is given by $\boldsymbol{T} = -p\boldsymbol{N}$, where $\boldsymbol{N}$ is the outer unit normal to the body and $p = -\rho_f g X_3$ is the hydrostatic pressure load.

a.) Show that the resultant surface force on the body is equal to the weight of the liquid displaced by the body in the positive X_3 direction (The Archimedes Principle).

b.) Show that the body has zero resultant torque about its centre of volume.

6.) A state of plane stress is given by $T_{31} = T_{32} = T_{33} = 0$.

a.) Explain why a change in Cartesian coordinates from $\boldsymbol{X}$ to $\widehat{\boldsymbol{X}}$ that leaves the X_3 direction unchanged (and preserves the handedness of the coordinate system) can be described by a single angle θ between the X_1 and $X_{\widehat{1}}$ directions.

b.) Find explicit relationships between the components of the Cauchy stress tensor in the two coordinate systems that involve only the angle θ.

c.) Find the values of θ for which $T_{\widehat{1}\widehat{2}} = 0$ and explain why these correspond to the maximum and minimum values of the normal stresses $T_{\widehat{1}\widehat{1}}$ and $T_{\widehat{2}\widehat{2}}$.

[1] Any feedback to: *Andrew.Hazel@manchester.ac.uk*

d.) Find the angle at which $T_{\hat{1}\hat{2}}$ attains its maximum value and show that it is $\pm 45°$ from the values of θ that extremise the normal stresses.

7.) By analogy with the strain tensor, find explicit expressions for the principal invariants of the Cauchy stress tensor, I_1, I_2 and I_3.

8.) The stress deviation tensor is defined by

$$\widetilde{T}^i_j = T^i_j - \frac{1}{3}T^k_k \delta^i_j.$$

Find explicit expressions for the principal invariants of $\widetilde{\mathsf{T}}$ which are conventionally denoted by J_1, J_2 and J_3 and show that the first invariant, J_1, is zero.

9.) The stress vector on a plane with unit normal $\boldsymbol{N}$ can be decomposed into normal and tangential (shear) directions so that

$$\boldsymbol{T} = T_{\boldsymbol{N}}\boldsymbol{N} + \boldsymbol{T}_S.$$

a.) Show that the normal stress is given by

$$T_{\boldsymbol{N}} = T_{IJ}N_I N_J,$$

where T_{IJ} and N_I are the components of the Cauchy stress tensor and the normal vector in a Cartesian coordinate system.

b.) Find an explicit expression for magnitude of the shear stress $|\boldsymbol{T}_S|^2$ in terms of the principal stresses and the components of the normal vector in the coordinate system given by the principal axes of stress. (The principal stresses are the eigenvalues of the mixed stress tensor and the principal axes of stress are the corresponding eigenvectors).

c.) The octahedral shear stress is the resultant shear stress on a plane that makes the same angle with each of three principal axes of the stress tensor. Show that the square of the magnitude of the octahedral shear stress is proportional to the second invariant of the stress deviation tensor J_2. **Hint:** You can use the formula derived above once you deduce the components of the normal to the required planes.

10.) A body occupying a region Ω has volume V and is subject to a body force $\boldsymbol{F}$ per unit volume and a traction field $\boldsymbol{\tau}$ acting on its surface, $\partial\Omega$.

a.) If the body is in equilibrium show that the average Cauchy stress is given by

$$\langle \mathsf{T} \rangle \equiv \frac{1}{V}\int_\Omega \mathsf{T}\,\mathrm{d}\mathcal{V} = \frac{1}{V}\left[\int_\Omega \boldsymbol{R}\otimes\boldsymbol{F}\,\mathrm{d}V + \int_{\partial\Omega}\boldsymbol{R}\otimes\boldsymbol{\tau}\,\mathrm{d}\mathcal{S}\right].$$

Hint: You will find it useful to first establish the identity

$$T_{IJ} = (X_J T_{IK})_{,K} + X_J F_I.$$

b.) If $\boldsymbol{F} = \mathbf{0}$ and the surface is loaded by a uniform pressure, p, show that

$$\langle \mathsf{T} \rangle = -p\,\mathsf{I},$$

where I is the identity. Further show that in this case $\mathsf{T} = \langle \mathsf{T} \rangle$ is a possible solution for an equilibrium state of the body.

11.) If A is a symmetric tensor find

$$\frac{D}{Dt}\left(\mathsf{A} : \mathsf{A}\right),$$

where the : indicates contraction of both indices, $\mathsf{A} : \mathsf{A} = A_{IJ}A_{IJ}$. You should express your answer in terms of A and $D\mathsf{A}/Dt$.

12.) Find the material derivative of the second Piola–Kirchhoff stress tensor $D\mathsf{S}/Dt$ in terms of the material derivative of the Cauchy stress tensor, the deformation gradient tensor and the Eulerian velocity gradient and rate of deformation tensors; and determine whether or not it is an objective tensor.

13.) The Truesdell rate for an objective tensor of second order, A is given by

$$\frac{D\mathsf{A}}{Dt} + \mathrm{tr}(\mathsf{D})\mathsf{A} + \mathsf{L}^T\mathsf{A} + \mathsf{A}\mathsf{L},$$

where, as usual, L is the Eulerian velocity gradient tensor and D is its symmetric part. Confirm that the Truesdell rate is objective.