MATH45061: EXAMPLE SHEET[1] II

1.) A deformation map $\boldsymbol{R} = \boldsymbol{\chi}(\boldsymbol{r})$ is defined in components in a global Cartesian basis by

$$X_1 = \mathrm{e}^{x_1}, \quad X_2 = 2x_2 - x_3, \quad X_3 = x_2 + 2x_3.$$

Find the deformation gradient tensor and determine whether the deformation is physically admissible for arbitrary initial domains. Sketch the deformed and undeformed bodies corresponding to a two-dimensional lamina for which $x_1, x_2 \in [0, 1]$ and $x_3 = 0$.

2.) An undeformed body in two-dimensions is described using the general coordinates

$$\boldsymbol{r} = \xi^1 \xi^2 \boldsymbol{e}_1 + \xi^2 \boldsymbol{e}_2,$$

where the base vectors $\boldsymbol{e}_i$ are with respect to a fixed Cartesian coordinate system. The body is moved into a deformed position given by $\boldsymbol{R} = 2\boldsymbol{r}$, represented in the alternative coordinates

$$\boldsymbol{R} = \chi^{\bar{1}} \boldsymbol{e}_1 + \chi^{\bar{1}} \chi^{\bar{2}} \boldsymbol{e}_2,$$

a.) Find the four possible sets of covariant base vectors:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \xi^i}, \quad \boldsymbol{g}_{\bar{i}} = \frac{\partial \boldsymbol{r}}{\partial \chi^{\bar{i}}}, \quad \boldsymbol{G}_i = \frac{\partial \boldsymbol{R}}{\partial \xi^i}, \quad \boldsymbol{G}_{\bar{i}} = \frac{\partial \boldsymbol{R}}{\partial \chi^{\bar{i}}}.$$

b.) Find the four associated metric tensors and compute the differences

$$A_{ij} = G_{ij} - g_{ij} \quad \text{and} \quad A_{\bar{i}\bar{j}} = G_{\bar{i}\bar{j}} - g_{\bar{i}\bar{j}}.$$

How are A_{ij} and $A_{\bar{i}\bar{j}}$ related?

3.) A region is deformed such that its position at time t is given by the components in a global Cartesian basis

$$X_1(t) = (1 + t)\, x_1, \quad X_2(t) = x_2 + t x_3 \quad \text{and} \quad X_3(t) = x_3 - t x_2,$$

where x_I represents the position of the body in the same coordinate system at $t = 0$.

a.) Prove that the deformation is physically admissible for all $t \geq 0$ and find expressions for $x_I(X_J)$.

b.) Find the velocity fields as functions of the Lagrangian and Eulerian coordinates, $\boldsymbol{v}(\boldsymbol{r}, t)$ and $\boldsymbol{V}(\boldsymbol{R}, t)$, respectively.

c.) Find the material derivative of the scalar function $\Phi(\boldsymbol{R}, t) = tX_1 + X_2$ by first converting to Lagrangian coordinates $\phi(\boldsymbol{r}, t) = \Phi(\boldsymbol{R}(\boldsymbol{r}, t), t)$.

d.) Find the material derivative of $\Phi(\boldsymbol{R}, t)$ directly using the formula derived in lectures. Does your answer agree with that found in part (c).

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4.) * **Hard** A surface in the deformed (Eulerian) configuration is defined by the expression $f(\chi^{\bar{i}}, t) = 0$, where $\chi^{\bar{i}}$ are arbitrary coordinates that are fixed in space. Prove that a necessary and sufficient condition for the surface to be a material surface (*i.e.* that it consists of the same material points) is that

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + V^{\bar{i}} \frac{\partial \phi}{\partial \chi^{\bar{i}}} = 0,$$

where $V^{\bar{i}}$ is the component of the velocity field in the direction of the covariant base vector $\boldsymbol{G}_{\bar{i}} = \partial \boldsymbol{R} / \partial \chi^{\bar{i}}$.

[**Hint:** For the necessary case, think about points in the surface moving with an arbitrary velocity and show that when that velocity coincides with the velocity of a material point the condition is true. For the sufficient case, use the method of characteristics for hyperbolic equations.]

5.) A rectangular block $-a \le x_1 \le a$, $-b \le x_2 \le b$, $-c \le x_3 \le c$ is deformed into a sector of the wall of a circular cylinder, so that the planes of constant x_1 become planes $R =$ constant; planes of constant x_2 become planes $\Theta =$ constant and planes of constant x_3 become those of constant Z in a cylindrical polar coordinate system (R, Θ, Z) in the deformed configuration.

a.) Explain why the deformation must be given by

$$R = f(x_1), \quad \Theta = g(x_2), \quad Z = h(x_3),$$

b.) In the case when $h(x_3) = \lambda x_3$, for a constant λ, find the covariant deformed metric tensor G_{ij}.

c.) If the deformation is isochoric (of constant volume) deduce that $\lambda f f' g' = 1$ and by using separation of variables find the functional forms of $f(x_1)$ and $g(x_2)$.

6.) Consider the pure dilation of a region of space such that the deformed position $\boldsymbol{R}$ of a material point $\boldsymbol{r}$ is given by

$$\boldsymbol{R} = \alpha \boldsymbol{r}.$$

a.) By expressing the deformation in Cartesian coordinates, find the Green–Lagrange strain tensor e_{IJ}.

b.) By expressing the deformation in spherical polar coordinates, find the strain tensor γ_{ij}.

7.) A unit vector $\boldsymbol{n} = n^i \boldsymbol{g}_i$, where the undeformed covariant basis $\boldsymbol{g}_j$ is associated with a general coordinate system ξ^i. Show that the equation

$$G_{ij} n^j - \mu g_{ij} n^j = 0,$$

can be written as

$$c_{IJ} n_J - \mu n_I = 0,$$

after transformation to a Cartesian coordinate system, so that $\boldsymbol{n} = n_I \boldsymbol{e}_I$.

Hence, find the principal stretches associated with the deformation in question 6 and confirm that they are invariant under change in coordinate system.

8.) Consider a simple shear of a block $x_I \in [0,1]$, where x_I are components in a global Cartesian basis $\boldsymbol{e}_I$,such that the deformed components in the same Cartesian basis are given by

$$X_1 = x_1, \quad X_2 = x_2 + \gamma x_3, \quad X_3 = x_3,$$

where $\gamma > 0$.

a.) Find the Green–Lagrange strain tensor e_{IJ}.

b.) Find the extreme values of the stretches associated with the deformation and the directions in which they occur.

c.) Find the shear (the change in angle) between two vectors originally directed along $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. Explain what happens to the shear in the limits $\gamma \to 0$ and $\gamma \to \infty$.

9.) Consider a one-dimensional bar, initially of unit length, that is initially stretched by λ_1 and subsequently by λ_2. We define a single coordinate direction $\boldsymbol{e}_1$ and the coordinate $x \in [0,1]$ describes the initial configuration of the bar $\boldsymbol{r} = x\,\boldsymbol{e}_1$.

a.) Explain why the final deformed position $X\,\boldsymbol{e}_1$ is given by

$$X = \lambda_2\,\lambda_1 x.$$

b.) Find the Green–Lagrange strain tensor associated with the first deformation. Find also the (incremental) strain tensor associated with the second deformation assuming that the reference configuration is that after the first deformation. For an incremental tensor, the Lagrangian coordinates should be based on the position after the first deformation. Finally, find the strain tensor associated with the total deformation from initial configuration to final configuration. Is there any simple relationship between the three strain tensors?

c.) Find the Hencky strain tensors, $\ln \mathsf{U}$, where U is the (right) stretch tensor, associated with the first, second (incremental) and total deformations. Is there any simple relationship between these three tensors?

10.) Consider the deformation given by

$$\begin{aligned} X_1 &= \cos(\omega t)\,x_1 + \sin(\omega t)\,x_2, \\ X_2 &= -\sin(\omega t)\,x_1 + \cos(\omega t)\,x_2, \\ X_3 &= (1 + \alpha t)\,x_3. \end{aligned}$$

a.) Find the Eulerian velocity gradient tensor, $V_{I,J}$.

b.) Find the Eulerian rate of deformation tensor and spin tensor and interpret your results.

11.) For a one-dimensional deformation prove that if the Eulerian rate of deformation tensor is zero, then the Eulerian rate of strain tensor must also be zero.

12.) Consider the deformation given by

$$\begin{aligned} X_1 &= \alpha\cos(\omega t)\,x_1 + \sin(\omega t)\,x_2, \\ X_2 &= -\alpha\sin(\omega t)\,x_1 + \cos(\omega t)\,x_2, \\ X_3 &= x_3. \end{aligned}$$

a.) Describe the deformation.

b.) Find the Eulerian rate of deformation tensor and explain why it does not coincide with the Eulerian rate of strain tensor unless $\alpha = \pm 1$.

13.) Show that the operation of twice the spin tensor acting on a vector can be written as the cross product between the vorticity and the vector, *i.e.*

$$2\mathsf{W} \cdot \boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v}.$$

Hence, show that the Eulerian acceleration can be written in the form

$$\boldsymbol{A} = \frac{D\boldsymbol{V}}{Dt} = \frac{\partial V}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{V} + \frac{1}{2}\nabla_{\boldsymbol{R}}|\boldsymbol{V}|^2.$$

14.) Show that the circulation Γ, the line integral of the parallel velocity along a closed path, is given by the integral of the vorticity over the area enclosed by the curve:

$$\Gamma = \int \boldsymbol{V} \cdot \mathbf{d}\boldsymbol{R} = \iint \boldsymbol{\omega} \cdot \mathbf{d}\boldsymbol{A}.$$

By considering the limit as the area tends to zero, give a physical interpretation of the vorticity.