MATH45061: EXAMPLE SHEET[1] I

1.) Two vectors are defined by

$$\boldsymbol{a} = \boldsymbol{e}_1 + 2\boldsymbol{e}_2 + 3\boldsymbol{e}_3, \quad \text{and} \quad \boldsymbol{b} = 4\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + \boldsymbol{e}_3,$$

where $\boldsymbol{e}_I$, $I = 1, 2, 3$ are the base vectors of a global Cartesian coordinate system. Write down the values of a^I and b_I and hence compute the scalar product $a^I b_I$.

2.) In a general coordinate system ξ^i, the position vector $\boldsymbol{r} = r^i \boldsymbol{g}_i$. Find an explicit expression for the components r^i. Are the components r^i equal to the coordinates in general? If ξ^i is chosen to be a Cartesian coordinate system, show that the components are equal to the coordinates.

3.) A coordinate system ξ^i is defined such that the position in a global two-dimensional Cartesian coordinate system (x, y) is given by

$$x = \xi^2 \xi^1, \quad y = \xi^1.$$

Find the covariant and contravariant base vectors.
Is the coordinate system orthogonal?

4.) Show that an orthonormal coordinate system can only be obtained by an orthogonal transformation from a fixed set of Cartesian base vectors. Hint: Start from the condition that the covariant and contravariant base vectors must be the same in an orthonormal coordinate system.

5.) A quantity $T(i, j)$ consists of nine coefficients defined so that

$$a^i = T(i, j) b^j, \quad \text{(summed over } j\text{)},$$

where $\boldsymbol{a} = a^i \boldsymbol{g}_i$ and $\boldsymbol{b} = b^j \boldsymbol{g}_j$ are both vectors. Show that $T(i, j)$ represent the components of a tensor T^i_j in the basis $\boldsymbol{g}_i \otimes \boldsymbol{g}^j$ provided that $T(i, j)$ is independent of $\boldsymbol{b}$.

6.) Find the metric tensors associated with the coordinate system given in question 3 and find their determinants. Verify that $\boldsymbol{g}^i = g^{ij} \boldsymbol{g}_j$.

7.) Establish the identity

$$\epsilon^{ijk} \epsilon_{klm} = g^i_l \, g^j_m - g^i_m \, g^j_l,$$

and hence establish the expansion of the vector triple product

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c}) \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b}) \boldsymbol{c}.$$

8.) Explain why the Christoffel symbol

$$\Gamma_{ijk} = \frac{\partial^2 \boldsymbol{r}}{\partial \xi^i \partial \xi^j} \cdot \frac{\partial \boldsymbol{r}}{\partial \xi^k},$$

is **not** a tensor.

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9.) By considering the covariant derivative of the scalar $\phi = a_j b^j$ show that the covariant derivative obeys the product rule.

10.) By considering the transformation properties of the (covariant or partial) derivative of the quantity $\alpha = A_{ij} b^i c^j$ where b^i and c^j are components of arbitrary vectors, determine the covariant derivative of the second-order tensor, $A_{ij}|_r$.

11.) Use the general formulæ established in lectures to calculate the gradient, divergence and Laplacian in cylindrical and spherical polar coordinates. Check that your results agree with those in standard references after converting the base vectors to the appropriate unit vectors where necessary.

12.) A general curve in space is given by $\boldsymbol{r}(t)$ for $t \in \mathbb{R}$. If a vector field $\boldsymbol{a}(\boldsymbol{r})$ is to remain parallel along the curve, explain why

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = 0,$$

and hence show that under these conditions all components of the covariant derivative with respect to a general coordinate system ξ^i must be zero, $a^i|_j = 0$. This gives an interpretation of the vanishing covariant derivative as providing a condition that ensures a vector field is parallel.