Chapter 7

Fluid Mechanics

Fluid mechanics includes the study of liquids and gases and all materials that exhibit fluidity: molecules can easily slip past each other so the materials flow (rather then simply deforming). Unlike solid materials, fluids cannot support shear stresses when not in motion, because the intermolecular attractions are relatively weak. Thus, when placed in a container, a fluid will flow until it reaches a static equilibrium position that depends on the shape of vessel (assuming that there is no forcing to induce continued motion). The arrangement of the individual material points within the equilibrium is not unique, however, and so a fluid does not have a unique natural rest state. The relative strength of intermolecular attractive forces is greater in a liquid than a gas, which means that a liquid has a finite minimum density (under isothermal conditions at least), whereas a gas will expand to completely fill the container and could, in theory at least, approach zero density as the container increases in size.

We expect that the material behaviour of a fluid will be affected by deformations that change the volume of material regions, equivalent to changes in density, as discussed in the section on ideal gases, §5.3. In addition, shear stresses do develop when the fluid is in motion, which can be interpreted as friction between molecules as they move past one another. Thus, we expect the stress to be a function of the rate of deformation of the fluid, $\dot{\mathsf{F}} = \mathsf{L}\mathsf{F}$ from equation 2.51), where $\mathsf{L} = \boldsymbol{\nabla}_{\!R} \otimes \boldsymbol{V}$ is the velocity gradient tensor[1] We have already seen in section 5.2.3, and specifically equations (5.3a, b), that only the symmetric part of the velocity gradient tensor, D, is objective. Hence, we should use D to represent the rate of deformation in order to ensure objective constitutive laws. The material properties of fluids are also affected by temperature and temperature gradients. Naturally, there are many other factors that we could include, but a wide class of behaviours is captured by the resulting constitutive assumptions

$$\Psi(\rho, \mathsf{D}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta), \quad \mathsf{T}(\rho, \mathsf{D}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta), \quad \eta(\rho, \mathsf{D}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta), \quad \boldsymbol{Q}(\rho, \mathsf{D}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta). \tag{7.1}$$

Once again, we shall consider the implications of the thermodynamic constraints on our proposed constitutive laws. The Clausius–Duhem inequality in free-energy form (4.29) is

$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!R}\Theta + \mathsf{T} : \mathsf{D} \geq 0.$$

Based on the form of the constitutive laws (6.1),

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\rho}\dot{\rho} + \frac{\partial\Psi}{\partial D_{IJ}}\dot{D}_{IJ} + \frac{\partial\Psi}{\partial\Theta}\dot{\Theta} + \frac{\partial\Psi}{\partial\Theta_{,I}}\dot{\Theta}_{,I},$$

[1]If you're confused by the dyadic notation, it can easily be converted into index notation $L_{IJ} = V_{J,I}$.

where tensor quantities have been represented in Cartesian coordinates for simplicity. Thus, the Clausius–Duhem inequality becomes

$$-\rho\frac{\partial\Psi}{\partial\rho}\dot{\rho} - \rho\frac{\partial\Psi}{\partial\mathsf{D}} : \dot{\mathsf{D}} - \rho\frac{\partial\Psi}{\partial\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\dot{\Theta} - \rho\left(\frac{\partial\Psi}{\partial\Theta} + \eta\right)\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta + \mathsf{T} : \mathsf{D} \geq 0,$$

which must be satisfied for all valid thermodynamic processes. Choosing a process for which the fluid is at rest and is subject to an instantaneous and uniform change in temperature, we find that

$$\left(\frac{\partial\Psi}{\partial\Theta} + \eta\right)\dot{\Theta} \leq 0.$$

The inequality must be true for positive and negative values of $\dot{\Theta}$ and from our constitutive assumptions (7.1) Ψ and η do not depend on $\dot{\Theta}$. Hence, exactly as thermoelastic solids,

$$\eta = -\frac{\partial\Psi}{\partial\Theta}. \tag{7.2}$$

By considering processes in which only $\dot{\mathsf{D}} \neq 0$ and then only $\boldsymbol{\nabla}_{\boldsymbol{R}}\dot{\Theta} \neq \mathbf{0}$, we obtain the additional constraints that

$$\frac{\partial\Psi}{\partial\mathsf{D}} = 0 \quad \text{and} \quad \frac{\partial\Psi}{\partial\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta} = 0.$$

In other words the Helmholtz free energy (and therefore the entropy) does not depend on the the velocity and temperature gradients,

$$\Psi(\rho, \Theta) \quad \text{and} \quad \eta(\rho, \Theta).$$

If we were able to construct an isothermal process for which $\dot{\rho} \neq 0$, but $\mathsf{D} = 0$, then the free energy would not depend on the density either. Such a process is not possible because, as we have already seen when considering ideal gases, D and $\dot{\rho}$ are related through the conservation of mass equation (5.7), $\dot{\rho} = -\rho\mathsf{I} : \mathsf{D}$. Using this relationship and the deduced constraints, the Clausius–Duhem inequality becomes

$$\left(\rho^2\frac{\partial\Psi}{\partial\rho}\mathsf{I} + \mathsf{T}\right) : \mathsf{D} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta \geq 0.$$

Unlike the ideal gas and the thermoelastic solid, the stress for a general fluid does depend on the rate of deformation, $\mathsf{T}(\rho, \mathsf{D}, \Theta, \boldsymbol{\nabla}_{\boldsymbol{R}}\Theta)$. Thus, we cannot vary D independently of T and it appears that we can make no further progress.

In fact, we can proceed by decomposing[2] the stress into two terms: (i) an "ideal gas like" term that determines the bulk response to density and temperature changes, but is independent of the rate of deformation and temperature gradients; and (ii) everything else:

$$\mathsf{T}(\rho, \mathsf{D}, \Theta, \boldsymbol{\nabla}_{\boldsymbol{R}}\Theta) = \mathsf{T}_0(\rho, \Theta) + \widetilde{\mathsf{T}}(\rho, \mathsf{D}, \Theta, \boldsymbol{\nabla}_{\boldsymbol{R}}\Theta), \tag{7.3}$$

where $\widetilde{\mathsf{T}}(\rho, 0, \Theta, 0) = 0$.

[2]Note that this additive decomposition has similarities with the means by which incompressibility constraints were imposed in §6.2.2. In the present context the decomposition is motivated by thermodynamic considerations, but the particular choice will be consistent with treating the pressure as a Lagrange multiplier that enforces the incompressibility constraint in fluids.

Returning to the Clausius–Duhem inequality for processes with zero temperature gradient and using the split (7.3), we obtain

$$\left(\rho^2 \frac{\partial \Psi}{\partial \rho} \mathsf{I} + \mathsf{T}_0 + \widetilde{\mathsf{T}}\right) : \mathsf{D} \geq 0.$$

We now consider a very slow deformations, so that $\mathsf{D} = \epsilon \widehat{\mathsf{D}}$, where $\widehat{\mathsf{D}} = \mathcal{O}(1)$, but $\epsilon \ll 1$. Taylor expansion of the stress about the state when $\mathsf{D} = 0$ gives

$$\mathsf{T} = \mathsf{T}_0 + \epsilon \frac{\partial \widetilde{\mathsf{T}}}{\partial \mathsf{D}} \widehat{\mathsf{D}} + \mathcal{O}(\epsilon^2).$$

We can therefore approximate the stress as

$$\mathsf{T} \approx \mathsf{T}_0 + \epsilon \widehat{\mathsf{T}},$$

where $\widehat{\mathsf{T}} = \frac{\partial \widetilde{\mathsf{T}}}{\partial \mathsf{D}} \widehat{D}$ and then the Clausius–Duhem inequality becomes

$$\epsilon \left(\rho^2 \frac{\partial \Psi}{\partial \rho} \mathsf{I} + \mathsf{T}_0\right) : \widehat{\mathsf{D}} + \epsilon^2 \widehat{\mathsf{T}} : \widehat{\mathsf{D}} + \mathcal{O}(\epsilon^3) \geq 0,$$

which means that in order for the inequality to be satisfied to leading order for any slow deformation

$$\mathsf{T}_0 = -\rho^2 \frac{\partial \Psi}{\partial \rho} \mathsf{I} = -P \mathsf{I},$$

where

$$P = \rho^2 \frac{\partial \Psi}{\partial \rho}, \tag{7.4}$$

is called the thermodynamic pressure and is consistent with the expression (5.9a) for the pressure of an ideal gas assuming that the temperature does not vary with density, which it does not.

The Clausius–Duhem inequality therefore reduces to

$$\widetilde{\mathsf{T}} : \mathsf{D} - \frac{1}{\Theta} \boldsymbol{Q} \cdot \boldsymbol{\nabla}_{\!\boldsymbol{R}} \Theta \geq 0.$$

If the stress depends on the temperature gradient and the heat flux depends on the rate of deformation then the expression cannot be simplified. If we make two final assumptions that

$$\widetilde{\mathsf{T}}(\rho, \mathsf{D}, \Theta) \quad \text{and} \quad \boldsymbol{Q}(\rho, \Theta, \boldsymbol{\nabla}_{\!\boldsymbol{R}} \Theta),$$

then the Clausius–Duhem inequality can be decomposed into two separate constraints

$$\widetilde{\mathsf{T}} : \mathsf{D} \geq 0 \quad \text{and} \quad -\frac{1}{\Theta} \boldsymbol{Q} \cdot \boldsymbol{\nabla}_{\!\boldsymbol{R}} \Theta \geq 0.$$

7.1 Isotropic Fluids: Newtonian and Reiner–Rivlin Fluids

From thermodynamic constraints, we have established that under our constitutive assumptions (7.1) the Helmholtz free energy and the entropy are functions only of density and temperature $\Psi(\rho, \Theta)$

and $\eta(\rho, \Theta)$, and that, under the additional constraint that it does not depend on temperature gradients, the Cauchy stress can be written in the form

$$\mathsf{T}(\rho, \Theta, \mathsf{D}) = -P(\rho, \Theta)\,\mathsf{I} + \widetilde{\mathsf{T}}(\rho, \Theta, \mathsf{D}).$$

If we also assume Fourier's law (6.7) for the heat flux then,

$$\boldsymbol{Q} = -\mathsf{K}(\rho, \Theta)\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta.$$

Unless the fluid contains internal structures, such as polymeric fluid or liquid crystals, it will be isotropic, in which case $\widetilde{\mathsf{T}}$ can only be a function of the invariants of D $({}_{\mathsf{D}}I_1,\ {}_{\mathsf{D}}I_2,\ {}_{\mathsf{D}}I_3)$ and the most general functional form is given by

$$\widetilde{\mathsf{T}} = \sum_n \alpha_n({}_{\mathsf{D}}I_1,\ {}_{\mathsf{D}}I_2,\ {}_{\mathsf{D}}I_3)\,\mathsf{D}^n,$$

which follows from the symmetry of both T and D. From the Cayley–Hamilton theorem, any matrix satisfies its own characteristic equation (which is simple to prove in the symmetric case by transforming to the eigenbasis), thus

$$-\mathsf{D}^3 + {}_{\mathsf{D}}I_1\mathsf{D}^2 - {}_{\mathsf{D}}I_2\mathsf{D} + {}_{\mathsf{D}}I_3\mathsf{I} = 0;$$

and all terms D^n for $n \geq 3$ can be expressed in terms of I, D, D^2 and the invariants. Hence,

$$\widetilde{\mathsf{T}} = \alpha_0\mathsf{I} + \alpha_1\mathsf{D} + \alpha_1\mathsf{D}^2,$$

where the dependence of $\alpha_{(i)}$ on the invariants has been suppressed for simplicity.

Thus, the full constitutive law is written as

$$\mathsf{T} = (-P + \alpha_0)\mathsf{I} + \alpha_1\mathsf{D} + \alpha_2\mathsf{D}^2, \tag{7.5}$$

which is the constitutive equation for a Reiner–Rivlin fluid. The invariant ${}_{\mathsf{D}}I_1$ is a linear function of D; ${}_{\mathsf{D}}I_2$ is quadratic; and ${}_{\mathsf{D}}I_3$ contains cubic terms, once the determinant is expanded out. Thus, if we wish to neglect any nonlinear behaviour, then $\alpha_2 = 0$, α_1 must be a constant and α_0 can depend only on ${}_{\mathsf{D}}I_1 = D^i_i = \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}$. We therefore arrive at the constitutive equations for a Newtonian fluid, in which the stress is linearly related to the rate of deformation and is conventionally written in the form[3]

$$\mathsf{T} = -P\,\mathsf{I} + \lambda(\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I} + 2\mu\mathsf{D}, \tag{7.6}$$

where λ is called the bulk modulus, or dilational viscosity, and μ is the dynamic viscosity. In general, λ and μ do vary with density and temperature, but they do not vary with the rate of deformation of the fluid.

[3]We can also arrive at this form by assuming the linear relationship $T^{\bar{i}\bar{j}} = -PG^{\bar{i}\bar{j}} + E^{\bar{i}\bar{j}\bar{k}\bar{l}}D_{\bar{k}\bar{l}}$, where, as in equation (6.43), the most general isotropic fourth-order tensor is given by

$$E^{\bar{i}\bar{j}\bar{k}\bar{l}} = \lambda G^{\bar{i}\bar{j}}G^{\bar{k}\bar{l}} + \mu\left(G^{\bar{i}\bar{k}}G^{\bar{j}\bar{l}} + G^{\bar{i}\bar{l}}G^{\bar{j}\bar{k}}\right),$$

which gives

$$T^{\bar{i}\bar{j}} = -PG^{\bar{i}\bar{j}} + \lambda G^{\bar{i}\bar{j}}D^{\bar{k}}_{k} + 2\mu D^{\bar{i}\bar{j}},$$

after using the symmetry properties of D.

7.1.1 Example: Rectilinear shear flow of a Reiner–Rivlin fluid

We wish to find the possible velocity profiles that a Reiner–Rivlin fluid can adopt when the velocity is only non-zero in one direction, but the profile varies with an orthogonal coordinate, *i.e.* $\boldsymbol{V} = H(Y)\,\boldsymbol{e}_X$. In index notation we would write

$$V_{\bar{1}} = H(X_2), \quad V_{\bar{2}} = 0 \quad V_{\bar{3}} = 0,$$

where the Eulerian coordinates $\chi^{\bar{i}}$ are taken to be the global Cartesian coordinates $X_{\bar{i}}$. We note that $\boldsymbol{\nabla_R}\boldsymbol{\cdot}\boldsymbol{V} = 0$, so the velocity field is isochoric (volume-preserving) and can therefore be imposed on both compressible and incompressible fluids.

The rate of deformation tensor is given by

$$D_{\bar{i}\bar{j}} = \frac{1}{2}\left(V_{\bar{i}}\|_{\bar{j}} + V_{\bar{j}}\|_{\bar{i}}\right) = \begin{pmatrix} 0 & \frac{1}{2}H' & 0 \\ \frac{1}{2}H' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $H'(Y) = \mathrm{d}H/\mathrm{d}Y$ because the covariant derivative is the partial derivative in Cartesian coordinates; and the partial derivative is actually an ordinary derivative because the only variations are with the single coordinate Y. Hence, from the constitutive law (7.5) the components of the Cauchy stress are

$$T_{\bar{i}\bar{j}} = (-P + \alpha_0)G_{\bar{i}\bar{j}} + \alpha_1 D_{\bar{i}\bar{j}} + \alpha_2 D_{\bar{i}\bar{k}} D^{\bar{k}}_{\bar{j}},$$

which becomes

$$T_{\bar{i}\bar{j}} = (-P + \alpha_0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\alpha_1\, H' \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}\alpha_2\,(H')^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Rightarrow \quad T_{\bar{i}\bar{j}} = \begin{pmatrix} -P + \alpha_0 + \frac{1}{4}\alpha_2(H')^2 & \frac{1}{2}\alpha_1 H' & 0 \\ \frac{1}{2}\alpha_1 H' & -P + \alpha_0 + \frac{1}{4}\alpha_2(H')^2 & 0 \\ 0 & 0 & -P + \alpha_0 \end{pmatrix}. \tag{7.7}$$

The coefficients α_0, α_1, α_2 are functions of the invariants of the rate of deformation tensor which are given by

$${}_{\mathsf{D}}I_1 = D^{\bar{i}}_{\bar{i}} = 0, \quad {}_{\mathsf{D}}I_2 = \frac{1}{2}\left[({}_{\mathsf{D}}I_1)^2 - D^{\bar{i}}_{\bar{j}} D^{\bar{j}}_{\bar{i}}\right] = -\frac{1}{4}(H')^2, \quad {}_{\mathsf{D}}I_3 = \det(\mathsf{D}) = 0,$$

and so $\alpha_i(H')$ and $\alpha_i(Y)$.

In Cartesian coordinates, all Christoffel symbols are zero, so Cauchy's equations (4.8) becomes

$$\rho\left[\frac{\partial V^{\bar{i}}}{\partial t} + V^{\bar{j}} V^{\bar{i}}_{,\bar{j}}\right] = T^{\bar{j}\bar{i}}_{,\bar{j}} + \rho F^{\bar{i}}.$$

The velocity field does not depend on time and is such that the convective acceleration, $V^{\bar{j}} V^{\bar{i}}_{,\bar{j}}$, is zero. Thus,

$$T^{\bar{j}\bar{i}}_{,\bar{j}} + \rho F^{\bar{i}} = 0,$$

and using the expression (7.7) for the stress, we obtain the governing equations

$$\frac{\partial}{\partial X}\left(-P+\alpha_0+\frac{1}{4}\alpha_2(H')^2\right)+\frac{\partial}{\partial Y}\left(\frac{1}{2}\alpha_1 H'\right)+\rho F^{\overline{1}}=0,$$
$$\frac{\partial}{\partial X}\left(\frac{1}{2}\alpha_1 H'\right)+\frac{\partial}{\partial Y}\left(-P+\alpha_0+\frac{1}{4}\alpha_2(H')^2\right)+\rho F^{\overline{2}}=0,$$
$$\frac{\partial}{\partial Z}\left(-P+\alpha_0\right)+\rho F^{\overline{3}}=0.$$

By definition $H(Y)$ and we have shown above that $\alpha_i(Y)$, so the equations are simply

$$-\frac{\partial P}{\partial X}+\frac{1}{2}\frac{\partial}{\partial Y}\left(\alpha_1 H'\right)+\rho F^{\overline{1}}=0, \tag{7.8a}$$
$$\frac{\partial}{\partial Y}\left(-P+\alpha_0+\frac{1}{4}\alpha_2(H')^2\right)+\rho F^{\overline{2}}=0, \tag{7.8b}$$
$$-\frac{\partial P}{\partial Z}+\rho F^{\overline{3}}=0. \tag{7.8c}$$

For a compressible fluid undergoing iosthermal, isochoric motion the pressure P remains constant because the density remains constant, see equation (7.4). Thus, the motion cannot support a body force in the Z direction. For simplicity, we shall set $\boldsymbol{F}=\boldsymbol{0}$, but simple body forces, such as gravity, in other directions can easily be included if desired. Integrating equation (7.8a) gives

$$\alpha_1 H' = C,$$

where C is a constant. We know that $\alpha_1(H')$, so excluding the degenerate[4] case when $\alpha_1 \propto (H')^{-1}$, the only possibility is that H' is a constant. When $H'=K$, a constant, all equations (7.8a–c) are satisfied and the velocity field is given by

$$V^{\overline{1}} = KY + A, \quad V^{\overline{2}} = V^{\overline{3}} = 0.$$

This velocity profile is sometimes called simple Couette flow and gives one solution for the motion of a fluid between two parallel plates that are translating relative to each other.

For an incompressible fluid, the pressure is an independent variable and the pressure variation in the Z direction can balance the Z-component of the body force. Nonetheless, we shall again set $\boldsymbol{F}=\boldsymbol{0}$, for simplicity. In this case, there is no pressure variation in the Z direction and $P(X,Y)$. Integrating equation (7.8b) gives

$$-P+\alpha_0+\frac{1}{4}\alpha_2(H')^2=-G(X),$$

for some unknown function $G(X)$. Equation (7.8a) will also be satisfied if

$$G'(X)=\frac{1}{2}\frac{\partial}{\partial Y}\left(\alpha_1 H'\right). \tag{7.9}$$

The left-hand side of equation (7.9) is a function only of X and the right-hand side is a function only of Y. It is only possible for the equation to be satisfied for all values of X and Y if both sides are equal to the same constant, say $-C$. Then

$$G(X)=-CX-E \quad \text{and} \quad \frac{1}{2}\alpha_1 H' = -CY+F,$$

[4]This case is clearly unphysical because it would give non-zero shear stresses $\overline{T}^{12}$ in uniform flow $H'=0$.

where E and F are constants of integration. Hence, the pressure is given by

$$P = \alpha_0 + \frac{1}{4}\alpha_2 (H')^2 - CX - E,$$

where the velocity profile H is given by the solution of the equation

$$\frac{1}{2}\alpha_1 H' = -CY + F = T^{\bar{1}\bar{2}}.$$

The equation may have to be solved numerically because particular choices of $\alpha_1(H')$ may make analytic integration rather awkward. Depending on the choice of α_1 we may have a number of different velocity profiles. For example, when α_1 is constant, the integration is straightforward and we obtain

$$V^{\bar{1}} = \frac{1}{\alpha_1}\left(2FY - CY^2\right),$$

which is the classic parabolic Poiseuille profile between two parallel plates that corresponds to a constant pressure gradient $\partial P/\partial X = -C$. We note that this profile is not possible for a compressible fluid because in that case a pressure gradient must induce a change in density.

If we chose $C = 0$ then, as in the compressible case, the solution is the Couette profile $H' = K = 2F/\alpha_1$. In this case, the normal stresses are given by

$$T^{\bar{1}\bar{1}} = T^{\bar{2}\bar{2}} = -P + \alpha_0 + \frac{1}{4}\alpha_2 K^2 = E, \quad T^{\bar{3}\bar{3}} = -P + \alpha_0,$$

which means that for a Reiner–Rivlin fluid with non-zero α_2 an additional normal stress (above the normal fluid pressure) must be applied in order to induce shear flow. Shear flows are typically generated in Couette devices, consisting of two parallel plates with the fluid between them. For a Newtonian fluid, linear shear flow can be induced by simply translating one plate in its own plane relative to the other. For more general Reiner–Rivlin fluids if $\alpha_2 > 0$ the normal stress differences will cause the two plates to move towards each other when the fluid is sheared. These "normal stress effects" (differences between the three normal stress components) are a feature of certain non-Newtonian fluids in shear flows. Particular examples include: (i) "rod climbing", also known as the Weissenberg effect, when a non-Newtonian fluid will rise up a rotating cylinder inserted into the fluid; and (ii) "die swell", when the fluid leaves an orifice.

7.1.2 Thermodynamic and Mechanical Pressure

The thermodynamic pressure P has already been defined in equation (7.4) to be $P = \rho^2 \partial\Psi/\partial\rho$. The mechanical pressure is simply $P_m = -\mathrm{trace}(\mathsf{T})/3$, which follows from the result that $\mathsf{T} = -P_m \mathsf{I}$ is the equilibrium stress field within a body loaded by only a constant constant external pressure[5] P_m.

From the constitutive law for a Newtonian fluid (7.6)

$$P_m = -\frac{1}{3}T^{\bar{i}}_{\bar{i}} = -\frac{1}{3}\left(-P\delta^{\bar{i}}_{\bar{i}} + \lambda(\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V})\,\delta^{\bar{i}}_{\bar{i}} + 2\mu D^{\bar{i}}_{\bar{i}}\right) = -\frac{1}{3}\left[-3P + (3\lambda + 2\mu)(\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V})\right],$$

because $\delta^{\bar{i}}_{\bar{i}} = 3$ and $D^{\bar{i}}_{\bar{i}} = \boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}$. Hence,

$$P_m = P - \left(\lambda + \frac{2}{3}\mu\right)(\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}), \tag{7.10}$$

[5] From continuity of stress over the entire boundary $T_{IJ}N_J = -P_m N_I$, which implies that $T_{IJ} = -P_m\delta_{IJ}$. The equilibrium equation $T_{IJ,J} = 0$ is trivially satisfied because P_m is a constant.

which means that the thermodynamic and mechanical pressures are only the same if the fluid is incompressible, or if $\lambda + \frac{2}{3}\mu = 0$, a condition known as the Stokes hypothesis. The coefficient $\lambda + \frac{2}{3}\mu = \mu_b$ is often called the bulk modulus of viscosity, or volume viscosity.

7.1.3 The Navier–Stokes equations

In components in the Eulerian coordinates, the balance of linear momentum is given by equation (4.8) and is

$$\rho\left[\frac{\partial V^{\bar{i}}}{\partial t} + V^{\bar{j}} V^{\bar{i}}||_{\bar{j}}\right] = T^{\bar{j}\,\bar{i}}||_{\bar{j}} + \rho F^{\bar{i}}.$$

In components in Eulerian coordinates, the Newtonian constitutive law (7.6) becomes

$$T^{\bar{i}\,\bar{j}} = -PG^{\bar{i}\,\bar{j}} + \lambda V^{\bar{k}}||_{\bar{k}} G^{\bar{i}\,\bar{j}} + 2\mu D^{\bar{i}\,\bar{j}}, \tag{7.11}$$

and from the definition of D, equation (2.50),

$$D^{\bar{i}\,\bar{j}} = \frac{1}{2}\left(G^{\bar{k}\,\bar{j}}\, V^{\bar{i}}||_{\bar{k}} + G^{\bar{k}\,\bar{i}}\, V^{\bar{j}}||_{\bar{k}}\right). \tag{7.12}$$

Using equation (7.12) in the constitutive law (7.11) and substituting the result into the balance of linear momentum, we obtain

$$\rho\left[\frac{\partial V^{\bar{i}}}{\partial t} + V^{\bar{j}} V^{\bar{i}}||_{\bar{j}}\right] = \left[-PG^{\bar{i}\,\bar{j}} + \lambda V^{\bar{k}}||_{\bar{k}} G^{\bar{i}\,\bar{j}} + \mu\left(G^{\bar{k}\,\bar{j}}\, V^{\bar{i}}||_{\bar{k}} + G^{\bar{k}\,\bar{i}}\, V^{\bar{j}}||_{\bar{k}}\right)\right]\Big|\Big|_{\bar{j}} + \rho F^{\bar{i}},$$

$$\Rightarrow \quad \rho\left[\frac{\partial V^{\bar{i}}}{\partial t} + V^{\bar{j}} V^{\bar{i}}||_{\bar{j}}\right] = G^{\bar{i}\,\bar{j}}\left(-P||_{\bar{j}} + \lambda V^{\bar{k}}||_{\bar{k}\,\bar{j}}\right) + \mu\left(G^{\bar{k}\,\bar{j}}\, V^{\bar{i}}||_{\bar{k}\,\bar{j}} + G^{\bar{k}\,\bar{i}}\, V^{\bar{j}}||_{\bar{k}\,\bar{j}}\right) + \rho F^{\bar{i}},$$

where we have assumed that μ and λ do not depend on space[6] and used the result[7] that $G^{\bar{i}\,\bar{j}}||_{\bar{j}} = 0$. In dyadic form, the equation becomes

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{V}\right] = -\nabla_{\boldsymbol{R}} P + (\lambda+\mu)\boldsymbol{\nabla}_{\boldsymbol{R}}\left(\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}\right) + \mu\boldsymbol{\nabla}^2_{\boldsymbol{R}}\boldsymbol{V} + \rho\boldsymbol{F}. \tag{7.13}$$

The energy equation in components in Eulerian coordinates is given by equation (4.18)

$$\rho\frac{D\Phi}{Dt} = T^{\bar{i}\,\bar{j}} D_{\bar{i}\,\bar{j}} + \rho B - Q^{\bar{i}}||_{\bar{i}}.$$

Using the constitutive equation (7.11) and Fourier's law in isotropic materials is $Q^{\bar{i}} = -\kappa G^{\bar{i}\,\bar{j}}\Theta||_{\bar{j}}$, we obtain

$$\rho\frac{D\Phi}{Dt} = \left[-PG^{\bar{i}\,\bar{j}} + \lambda V^{\bar{k}}||_{\bar{k}} G^{\bar{i}\,\bar{j}} + 2\mu D^{\bar{i}\,\bar{j}}\right] D_{\bar{i}\,\bar{j}} + \rho B + \kappa G^{\bar{i}\,\bar{j}}\Theta||_{\bar{j}\,\bar{i}},$$

$$\Rightarrow \quad \rho\left[\frac{\partial\Phi}{\partial t} + V^{\bar{i}}\Phi||_{\bar{i}}\right] = \left(-P + \lambda V^{\bar{k}}_{\bar{k}}\right) D^{\bar{i}}_{\bar{i}} + 2\mu D^{\bar{i}\,\bar{j}} D_{\bar{i}\,\bar{j}} + \rho B + \kappa G^{\bar{i}\,\bar{j}}\Theta||_{\bar{i}\,\bar{j}}.$$

---

[6]This is not generally the case when we take temperature variation of μ and λ into account.

[7]This result is easily proved by transforming into Cartesian coordinates, in which $G^{\bar{i}\,\bar{j}}||_{\bar{j}} = \delta_{IJ,J} = 0$.

This can be written in dyadic form as

$$\rho\left[\frac{\partial\Phi}{\partial t}+\boldsymbol{V}\cdot\boldsymbol{\nabla}\Phi\right]=-P\,\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}+\lambda\,(\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V})^2+2\mu\mathsf{D}:\mathsf{D}+\rho B+\kappa\boldsymbol{\nabla}_{\boldsymbol{R}}^2\Theta. \tag{7.14}$$

In addition, mass must be conserved, so the governing equations for a compressible Newtonian fluid are

$$\frac{D\rho}{Dt}+\rho\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}=0, \tag{7.15a}$$

$$\rho\frac{D\boldsymbol{V}}{Dt}+\boldsymbol{\nabla}_{\boldsymbol{R}}\,P-\rho\boldsymbol{F}=(\lambda+\mu)\boldsymbol{\nabla}_{\boldsymbol{R}}\,(\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V})+\mu\boldsymbol{\nabla}_{\boldsymbol{R}}^2\boldsymbol{V}, \tag{7.15b}$$

$$\rho\frac{D\Phi}{Dt}+P\,\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}-\rho B=\lambda\,(\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V})^2+2\mu\mathsf{D}:\mathsf{D}+\kappa\boldsymbol{\nabla}_{\boldsymbol{R}}^2\Theta. \tag{7.15c}$$

These are often called the Navier–Stokes equations, although there is a certain ambiguity about exactly which combination of equations is included, but it is, at least, the balance of momentum equation. On comparison of the equations (7.15a–c) with the ideal gas equations (5.15a–c), we note that the mass conservation is unchanged because it is entirely kinematic, but that dissipative mechanisms have been introduced into the other two equations, via the terms multiplied by λ, μ and κ. These terms can be interpreted as representing the net exchange of momentum between molecules in dilation (λ) and shear (μ) and the transfer of heat energy between molecules (κ). For a given motion, as μ and λ increase the amount of momentum lost in molecular collisions also increases and therefore greater forcing is required to achieve the same acceleration. Thus, fluids with higher values of μ and λ have greater resistance to motion.

If the fluid is incompressible than $\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}=0$ and then governing equations reduce to the incompressible Navier–Stokes equations and the energy equation

$$\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}=0, \tag{7.16a}$$

$$\rho\frac{D\boldsymbol{V}}{Dt}=-\boldsymbol{\nabla}_{\boldsymbol{R}}\,P+\mu\boldsymbol{\nabla}_{\boldsymbol{R}}^2\boldsymbol{V}+\rho\boldsymbol{F}, \tag{7.16b}$$

$$\rho\frac{D\Phi}{Dt}=+2\mu\mathsf{D}:\mathsf{D}+\kappa\boldsymbol{\nabla}_{\boldsymbol{R}}^2\Theta+\rho B. \tag{7.16c}$$

7.1.4 Dimensional analysis and the Reynolds number

If we consider an isothermal, incompressible Newtonian fluid, then the internal energy $\Phi(\rho,\Theta)$ cannot change because the temperature and density remain fixed. Hence the energy equation (7.16c) becomes

$$2\mu\mathsf{D}:\mathsf{D}+\rho B=0,$$

which means that any energy generated by viscous terms must be removed from the system by an external heat sink ($\mathsf{D}:\mathsf{D}\geq 0$ and $\mu\geq 0$ from thermodynamic constraints). We can simply assume that a "magic" heat sink exists that always extracts just the right amount of heat. Alternatively, as is often the case in practice, we shall assume that the heat generated by viscous effects is relatively small, so that the heat quickly diffuses back to a give a uniform temperature state.

In the absence of thermal effects and body forces, the governing equations for the flow of an incompressible Newtonian fluid reduce to

$$\boldsymbol{\nabla}_{\boldsymbol{R}}\cdot\boldsymbol{V}=0,$$

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} + \rho\boldsymbol{V}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{V} = -\boldsymbol{\nabla}_{\boldsymbol{R}}\, P + \mu\boldsymbol{\nabla}_{\boldsymbol{R}}^2\boldsymbol{V}.$$

We can consider the relative importance of each of the terms by using dimensional analysis. The idea is extremely simple, but very powerful. We introduce typical time, length, velocity and pressure scales, $\mathcal{T}$, $\mathcal{L}$, $\mathcal{U}$ and $\mathcal{P}$, respectively, and write

$$t = \mathcal{T}t^*, \quad \boldsymbol{R} = \mathcal{L}\boldsymbol{R}^*, \quad \boldsymbol{V} = \mathcal{U}\boldsymbol{V}^*, \quad P = \mathcal{P}P^*. \tag{7.17}$$

The quantities t^*, $\boldsymbol{R}^*$, $\boldsymbol{V}^*$ and P^* are all dimensionless because the dimensions are contained within the chosen scales. For example, if we have a dimensional position $X = 10$ metres and we choose one length scale to be $\mathcal{L}_1 = 1$ metre and another to be $\mathcal{L}_2 = 1$ millimetre, we can write

$$\begin{array}{ccccc} X & = & 10 & \mathcal{L}_1, \\ \text{[metres]} & = & & \text{[metres]}, \end{array} \qquad \begin{array}{ccccc} X & = & 10000 & \mathcal{L}_2, \\ \text{[metres]} & = & & \text{[millimetres]}; \end{array}$$

and the numerical multipliers (X^*) are dimensionless.

Using the non-dimensionalisation defined by equations (7.17) in the governing equations, we obtain

$$\frac{\mathcal{U}}{\mathcal{L}}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{\cdot}\boldsymbol{V}^* = 0,$$

$$\rho\frac{\mathcal{U}}{\mathcal{T}}\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \rho\frac{\mathcal{U}^2}{\mathcal{L}}\boldsymbol{V}^*\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{V}^* = -\frac{\mathcal{P}}{\mathcal{L}}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\, P^* + \frac{\mu\mathcal{U}}{\mathcal{L}^2}\boldsymbol{\nabla}_{\boldsymbol{R}^*}^2\boldsymbol{V}^*.$$

In order to obtain dimensionless equations, we divide by the dimensional groupings that multiply each term. In order for the equations to make physical sense in the first place, the dimensions of each term must be the same, so we can divide by any of the groupings.

The dimensionless conservation of mass equation is essentially unchanged from the dimensional version,

$$\boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{\cdot}\boldsymbol{V}^* = 0.$$

We divide all terms in the linear momentum equation by the dimensional grouping $\mu\mathcal{U}/\mathcal{L}^2$ to obtain

$$\frac{\rho\mathcal{L}^2}{\mu\mathcal{T}}\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \frac{\rho\mathcal{U}\mathcal{L}}{\mu}\boldsymbol{V}^*\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{V}^* = -\frac{\mathcal{P}\mathcal{L}}{\mu\mathcal{U}}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\, P^* + \boldsymbol{\nabla}_{\boldsymbol{R}^*}^2\boldsymbol{V}^*.$$

The equation is now in dimensionless form and, in fact, we can extract a common dimensionless factor on the left-hand side,

$$\frac{\rho\mathcal{U}\mathcal{L}}{\mu}\left[\frac{\mathcal{L}}{\mathcal{U}\mathcal{T}}\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^*\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{V}^*\right] = -\frac{\mathcal{P}\mathcal{L}}{\mu\mathcal{U}}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\, P^* + \boldsymbol{\nabla}_{\boldsymbol{R}^*}^2\boldsymbol{V}^*.$$

It is conventional to assign names to each dimensionless grouping, we write

$$Re\left[St\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^*\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{V}^*\right] = -\frac{\mathcal{P}\mathcal{L}}{\mu\mathcal{U}}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\, P^* + \boldsymbol{\nabla}_{\boldsymbol{R}^*}^2\boldsymbol{V}^*, \tag{7.18}$$

$$\text{where} \quad Re = \frac{\rho\mathcal{U}\mathcal{L}}{\mu} \quad \text{and} \quad St = \frac{\mathcal{L}}{\mathcal{U}\mathcal{T}}.$$

The quantity Re is called the Reynolds number and expresses the relative importance of fluid inertia to viscous forces. The quantity St is called the Strouhal number and represents the ratio of the natural timescale of the fluid motion $\mathcal{L}/\mathcal{U}$ to our chosen timescale $\mathcal{T}$. Unless there is an external timescale, such as a periodic forcing, we would normally choose $\mathcal{T} = \mathcal{L}/\mathcal{U}$, so that $St = 1$.

As $Re \to 0$, fluid viscosity is the dominant force and the equations reduce to the (linear) Stokes equations (provided that St remains $\mathcal{O}(1)$)

$$-\frac{\mathcal{PL}}{\mu\mathcal{U}}\boldsymbol{\nabla}_{\boldsymbol{R}^*} P^* + \boldsymbol{\nabla}^2_{\boldsymbol{R}^*}\boldsymbol{V}^* = 0.$$

In this limit, it is sensible to choose the pressure scaling such that

$$\frac{\mathcal{PL}}{\mu\mathcal{U}} = 1 \quad \Rightarrow \quad \mathcal{P} = \frac{\mu\mathcal{U}}{\mathcal{L}},$$

which means that the pressure is scaled on the viscous scale, and then the Stokes equations become

$$\boldsymbol{\nabla}_{\boldsymbol{R}^*} P^* = \boldsymbol{\nabla}^2_{\boldsymbol{R}^*}\boldsymbol{V}^* \quad \text{and} \quad \boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{\cdot}\boldsymbol{V}^* = 0. \tag{7.19}$$

The physical interpretation of these equations is rather simple, the only forces present are the fluid pressure and viscosity. Applying a pressure gradient will tend to accelerate the fluid, but viscous forces will resist the acceleration. In order to have steady flow, the pressure gradient must exactly balance the viscous forces so that there is no acceleration.

Returning to dimensionless Navier–Stokes equations (7.18),

$$Re\left[St\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^*\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{V}^*\right] = -\frac{\mathcal{PL}}{\mu\mathcal{U}}\boldsymbol{\nabla}_{\boldsymbol{R}^*} P^* + \boldsymbol{\nabla}^2_{\boldsymbol{R}^*}\boldsymbol{V}^*,$$

we can also consider the limit as $Re \to \infty$, which is the case when inertia dominates. If we retain our viscous scaling for the pressure then dividing by Re and taking the limit gives

$$St\,\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^*\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{V}^* = \boldsymbol{0}, \tag{7.20}$$

which is known as Burger's equation. This seems a bit strange, however, because we should expect to recover the linear momentum equation of the ideal gas by taking the limit $\mu \to 0$, which corresponds to $Re \to \infty$, so what has gone wrong? The answer is that when we take the limit $\mu \to 0$, the viscous pressure scale $\mathcal{P} = \mu\mathcal{U}/\mathcal{L} \to 0$, which means that we scale the pressure out of the problem. If we have no pressure gradient then we have no acceleration and when we consider steady flow $\partial\boldsymbol{V}^*/\partial t^* = \boldsymbol{0}$, then the only solution of equation (7.20) is one for which $\boldsymbol{V}^*\boldsymbol{\cdot}\boldsymbol{V}^* = C$, where C is a constant.

Physically, we expect pressure gradients to be important even when there is no viscosity (we know this from the dimensional equations), and the solution to our problem is to choose our pressure scale such that

$$Re = \frac{\mathcal{PL}}{\mu\mathcal{U}} \quad \Rightarrow \quad \mathcal{P} = \frac{\mu\mathcal{U}}{\mathcal{L}}\frac{\rho\mathcal{UL}}{\mu} = \rho\mathcal{U}^2,$$

which means that the pressure is now scaled on the inertial scale. Having rescaled the pressure in this way, dividing equation (7.18) by Re and taking the limit as $Re \to \infty$, does indeed recover the linear momentum equation associated with an ideal gas, albeit in dimensionless form,

$$St\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^*\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}^*}\boldsymbol{V}^* = -\boldsymbol{\nabla}_{\boldsymbol{R}^*} P^*. \tag{7.21}$$

This equation is also known as the Euler equation and represents a simple balance between acceleration, fluid inertia (force due to the fact that the fluid is already moving) and pressure gradients.

7.1.5 Boundary conditions

The boundary conditions that should be applied to a fluid remain a matter of some debate in certain cases, particularly in the vicinity of fluid interfaces. In general, however, we may specify a boundary velocity or a boundary traction, or a combination of the two such as the Navier slip condition

$$\boldsymbol{V} - \boldsymbol{V}_{\text{wall}} = \eta \mathsf{T} \cdot \boldsymbol{N},$$

where $\boldsymbol{V}_{\text{wall}}$ is the velocity of the boundary and η is known as the slip coefficient. In the vast majority of cases, experiments have demonstrated that when a fluid is adjacent to a solid boundary it satisfies a no-slip condition $\eta = 0$, in which case

$$\boldsymbol{V} = \boldsymbol{V}_{\text{wall}}.$$

Traction boundary conditions are less common, but flat interfaces can be approximated by a traction-free condition.

In addition, for the energy equation we can specify a temperature, or a heat flux (temperature gradient) or, again, a combination of the two. The combined boundary conditions usually express reaction of some form at the surface.

Finally, for unsteady problems, we must also supply an initial state. If the initial state is inconsistent with the boundary conditions, then this will usually manifest as a singularity of some form or another. Investigation of singularities in fluid mechanics is, in fact, a field of study in its own right.

7.1.6 The concept of the boundary layer

The equation (7.21) is first order in space (only has single derivatives of the form $\partial/\partial X^*$), unlike the momentum balance in the Navier–Stokes equations (7.16b), which is second order in space. The loss of the highest spatial derivative is a singular perturbation to the equation because fewer boundary conditions can be applied. In fact, for the Euler equations, we can only specify the normal velocity[8] to the surface, which means that we cannot enforce the no-slip condition. Thus, an inviscid (ideal) fluid has no means of imposing drag on a solid body, a result known as D'Alembert's paradox. The resolution of the paradox is to recognise that even for very large values of the Reynolds number, the viscous term in the Navier–Stokes equations can remain $\mathcal{O}(1)$ for sufficiently large values of $\boldsymbol{\nabla}^2_{\boldsymbol{R}^*} \boldsymbol{V}^*$ (if they are $\mathcal{O}(Re)$ in fact). If a no-slip boundary condition is enforced, then the velocity must decrease from the value in the "free stream" to the value of zero over a short distance, which gives a sufficiently large value of the velocity variations. The region over which the adjustment takes place is called a viscous boundary layer.

These arguments can be made formal by considering the boundary layer that develops when an incompressible, Newtonian fluid of uniform velocity flows past a stationary flat wall. We start with the dimensionless steady, two-dimensional, incompressible Navier–Stokes equations:

$$U^* \frac{\partial U^*}{\partial X^*} + V^* \frac{\partial U^*}{\partial Y^*} = -\frac{\partial P^*}{\partial X^*} + \frac{1}{Re}\left(\frac{\partial^2 U^*}{\partial X^{*2}} + \frac{\partial^2 U^*}{\partial Y^{*2}}\right), \tag{7.22a}$$

$$U^* \frac{\partial V^*}{\partial X^*} + V^* \frac{\partial V^*}{\partial Y^*} = -\frac{\partial P^*}{\partial Y^*} + \frac{1}{Re}\left(\frac{\partial^2 V^*}{\partial X^{*2}} + \frac{\partial^2 V^*}{\partial Y^{*2}}\right), \tag{7.22b}$$

$$\frac{\partial U^*}{\partial X^*} + \frac{\partial V^*}{\partial Y^*} = 0, \tag{7.22c}$$

[8] Actually, we can only specify one component of the velocity and for an ideal fluid there is no mechanism to impart tangential velocity from a surface (no shear stresses), so the natural condition is to impose the normal velocity.

where the dimensionless position $\boldsymbol{R}^* = (X^*, Y^*)$ and velocity $\boldsymbol{V}^* = (U^*, V^*)$ in global Cartesian coordinates. We assume that the wall is located at $Y^* = 0$ and that the fluid occupies the region $-\infty < X^* < \infty$ and $Y^* \geq 0$. The velocity in the "free stream" far away from the wall is given by $\boldsymbol{V}^* = (U^*_{fs}(X^*), 0)$, provided that we have chosen our velocity scale $\mathcal{U}$ to be of the same order of magnitude as the "free stream" velocity; and the boundary condition on the wall is that $\boldsymbol{V}^* = \boldsymbol{0}$ at $Y^* = 0$.

We assume that the boundary layer is thin and we rescale into boundary-layer coordinates $(\widetilde{X}, \widetilde{Y})$ such that $X^* = \widetilde{X}$ and $Y^* = \delta\widetilde{Y}$, where $\delta \ll 1$. Note that this scaling ensures that $Y^* \ll X^*$ for $\mathcal{O}(1)$ values of the boundary-layer coordinates. We rescale the velocities $U^* = \widetilde{U}$ and $V^* = \delta\widetilde{V}$, so that both terms remain in balance in the continuity equation

$$\frac{\partial \widetilde{U}}{\partial \widetilde{X}} + \frac{\delta}{\delta}\frac{\partial \widetilde{V}}{\partial \widetilde{Y}} = 0 \quad \Rightarrow \quad \frac{\partial \widetilde{U}}{\partial \widetilde{X}} + \frac{\partial \widetilde{V}}{\partial \widetilde{Y}} = 0.$$

Thus, in order to conserve mass, the velocity normal to the boundary must be much smaller than the tangential velocity, which simply reflects the fact that layer is thin. We assume that the pressure remains of the same order, $\widetilde{P} = P^*$, and then the momentum equations become

$$\widetilde{U}\frac{\partial \widetilde{U}}{\partial \widetilde{X}} + \frac{\delta}{\delta}\widetilde{V}\frac{\partial \widetilde{U}}{\partial \widetilde{Y}} = \widetilde{U}\frac{\partial \widetilde{U}}{\partial \widetilde{X}} + \widetilde{V}\frac{\partial \widetilde{U}}{\partial \widetilde{Y}} = -\frac{\partial \widetilde{P}}{\partial \widetilde{X}} + \frac{1}{Re}\frac{\partial^2 \widetilde{U}}{\partial \widetilde{X}^2} + \frac{1}{\delta^2\, Re}\frac{\partial^2 \widetilde{U}}{\partial \widetilde{Y}^2}, \tag{7.23a}$$

$$\delta\widetilde{U}\frac{\partial \widetilde{V}}{\partial \widetilde{X}} + \frac{\delta^2}{\delta}\widetilde{V}\frac{\partial \widetilde{V}}{\partial \widetilde{Y}} = \delta\left[\widetilde{U}\frac{\partial \widetilde{V}}{\partial \widetilde{X}} + \widetilde{V}\frac{\partial \widetilde{V}}{\partial \widetilde{Y}}\right] = -\frac{1}{\delta}\frac{\partial \widetilde{P}}{\partial \widetilde{Y}} + \frac{\delta}{Re}\frac{\partial^2 \widetilde{V}}{\partial \widetilde{X}^2} + \frac{\delta}{\delta^2 Re}\frac{\partial^2 \widetilde{V}}{\partial \widetilde{Y}^2}. \tag{7.23b}$$

We can retain a viscous (second-order) term in equation (7.23a) as $Re \to \infty$ if we choose $\delta^2 = Re^{-1}$, *i.e.* $\delta = Re^{-1/2}$ — the classic boundary-layer scaling. In the limit $Re \to \infty$, under this assumption, we have the boundary-layer equations

$$\widetilde{U}\frac{\partial \widetilde{U}}{\partial \widetilde{X}} + \widetilde{V}\frac{\partial \widetilde{U}}{\partial \widetilde{Y}} = -\frac{\partial \widetilde{P}}{\partial \widetilde{X}} + \frac{\partial^2 \widetilde{U}}{\partial \widetilde{Y}^2}, \tag{7.24a}$$

$$\frac{\partial \widetilde{P}}{\partial \widetilde{Y}} = 0. \tag{7.24b}$$

The second equation follows after multiplying equation (7.23b) by δ and taking the limit $Re \to \infty$; and we see that the pressure does not vary through the thickness of the boundary layer. The pressure can therefore be found by taking the pressure at the outer edge of the boundary layer when we reach the "free stream". At the outer edge, we have uniform inviscid flow such $\boldsymbol{V}^* = (U^*_{fs}(X^*), 0) = (U^*_{fs}(\widetilde{X}), 0)$ and so

$$U^*_{fs}\frac{\partial U^*_{fs}}{\partial \widetilde{X}} = -\frac{\partial \widetilde{P}}{\partial \widetilde{X}},$$

which means that the boundary-layer equations becomes

$$\widetilde{U}\frac{\partial \widetilde{U}}{\partial \widetilde{X}} + \widetilde{V}\frac{\partial \widetilde{U}}{\partial \widetilde{Y}} = U^*_{fs}\frac{\partial U^*_{fs}}{\partial \widetilde{X}} + \frac{\partial^2 \widetilde{U}}{\partial \widetilde{Y}^2}, \tag{7.25a}$$

$$\frac{\partial \widetilde{U}}{\partial \widetilde{X}} + \frac{\partial \widetilde{V}}{\partial \widetilde{Y}} = 0. \tag{7.25b}$$

The boundary conditions are those of no slip on the wall ($\widetilde{Y} = 0$), $\widetilde{U} = 0$, $\widetilde{V} = 0$; and far away from the wall ($\widetilde{Y} \to \infty$) the velocity matches to the "free stream" $\widetilde{U} \to U^*_{fs}$, $\widetilde{V} \to 0$.

The boundary-layer equations, and the many variations thereof, have been the subject of intense study for over a century since their discovery by Prandtl in 1904. The entire body of work devoted to the subject is called boundary-layer theory.