Three hours

THE UNIVERSITY OF MANCHESTER

CONTINUUM MECHANICS

25 January 2018

14:00 – 17:00

Answer **ALL THREE** questions in section A (21 marks in total).
Answer **THREE** of the **FOUR** questions in section B (54 marks in total). If more than THREE questions from Section B are attempted, then credit will be given for the best THREE answers.

University approved calculators may be used.

SECTION A

A1. A continuous body is deformed such that at time t its position in a global Cartesian coordinate system is given by

$$X_1 = x_1 + a^2 t^2 x_2, \quad X_2 = (1 + a^2 t^2) x_2, \quad X_3 = x_3,$$

where a is a constant and x_I are the coordinates of material points at $t = 0$.

(i) Find the velocity of the body from the Lagrangian viewpoint, $v(x_1, x_2, x_3, t)$.

(ii) Find the velocity of the body from the Eulerian viewpoint, $V(X_1, X_2, X_3, t)$.

(iii) Show that a material plane initially defined by $x_1 = b$ deforms into another plane at time t with (non-unit) normal given by

$$\boldsymbol{N} = \left(1, \ \frac{-a^2 t^2}{1 + a^2 t^2}, \ 0\right).$$

You may use the result that a plane is defined by $\boldsymbol{X} \cdot \boldsymbol{N} = c$, where c is a constant, without proof.

[6 marks]

A2. Small charged particles are advected by a moving continuum, where $N(\boldsymbol{R}, t)$ denotes the number of particles per unit deformed volume, $\boldsymbol{R}$ is the Eulerian position and t is time.

(i) If each particle carries a constant charge q and the total charge within a material region Ω_t is conserved, show that this condition is described by the equation

$$\frac{\partial N}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot (N \boldsymbol{V}) = 0,$$

where $\boldsymbol{V}$ is the velocity of the continuum.

(ii) Derive the modified conservation equation when there is a distributed source of charge, $\mathcal{Q}(\boldsymbol{R}, t)$, within the continuum. Be sure to specify the units of $\mathcal{Q}$ carefully.

[7 marks]

A3. A vector $\boldsymbol{b}$ in the undeformed configuration of a continuum is mapped to the vector $\boldsymbol{B} = \mathsf{F}\boldsymbol{b}$ via the deformation gradient tensor F, where $F_{IJ} = \partial X_I / \partial x_J$ in the usual notation.

(i) By considering the material derivative of $\boldsymbol{B}$, or otherwise, explain why the material derivative is not an objective rate.

(ii) Show that the upper convected derivative

$$\boldsymbol{B}^{\nabla} = \frac{D\boldsymbol{B}}{Dt} - \mathsf{L}\boldsymbol{B},$$

is objective, where L is the Eulerian velocity gradient tensor, $L_{IJ} = \partial V_I / \partial X_J$.

[8 marks]

SECTION B

B4. Consider a cylindrical tube of hyperelastic material with undeformed inner radius c, undeformed outer radius d and and undeformed length l. The tube is deformed in such a way that it is not twisted and the new inner radius is C, the new outer radius is D and the new length is L.

(i) If a cylindrical polar coordinate system (r, θ, z) is defined in the undeformed configuration, explain under what further assumptions the deformed position can be written in the form

$$X_1 = R(r)\cos\theta, \quad X_2 = R(r)\sin\theta, \quad X_3 = \lambda z,$$

where λ is a positive constant and $R(r)$ is an unknown function.

(ii) Calculate the strain invariants I_1, I_2, I_3.

(iii) If the material is incompressible, find an explicit expression for $R(r)$ and hence find expressions for C and D in terms of c, d, l and L.

(iv) Find an explicit expression for the components of the stress tensor T^{ij} in terms of a pressure P, λ and the derivatives of the strain energy function with respect to the strain invariants: *i. e.* the quantities A and B in the formula sheet.

[18 marks]

B5. A second-order incompressible fluid has Cauchy stress given by

$$\mathsf{T} = \left\{a_1 + a_2 \text{trace}\left[\left(\mathsf{A}^{(1)}\right)^2\right]\right\}\mathsf{A}^{(1)} + a_3\left(\mathsf{A}^{(1)}\right)^2,$$

where the quantities a_i are constants; and the Rivlin–Ericksen tensor of order m, $\mathsf{A}^{(m)}$, is defined by

$$\frac{D^m}{Dt^m}|\mathrm{d}\boldsymbol{R}|^2 = \mathsf{A}^{(m)}_{\bar{i}\bar{j}}\,\mathrm{d}\chi^{\bar{i}}\,\mathrm{d}\chi^{\bar{j}},$$

where $\chi^{\bar{i}}$ are general Eulerian coordinates.

(i) Show that $\mathsf{A}^{(1)} = \mathsf{L} + \mathsf{L}^T$, where L the Eulerian velocity gradient tensor, i.e. $L_{ij} = V_i||_j$ in the usual notation.

(ii) The fluid flows steadily through a rigid cylindrical pipe of radius R under the action of an axial body force $\rho\boldsymbol{F} = G\boldsymbol{e}_z$, where G is a constant and $\boldsymbol{e}_z$ is the unit vector directed along the cylinder axis. Assuming that the fluid velocity is in the axial direction, remains bounded and is a function only of the radial position within the tube, r, find an explicit expression for the velocity in the Eulerian viewpoint.

(iii) Show that if $a_3 \neq 0$, there must also be a radial body force in order for Cauchy's equation to be satisfied.

[18 marks]

Hint: In cylindrical polar coordinates $(\xi^1, \xi^2, \xi^3) = (r, \theta, z)$ the only non-zero Christoffel symbols are given by

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \quad \text{and} \quad \Gamma^1_{22} = -r.$$

B6. A magnetoelastic material responds to electromagnetic and mechanical forcing and is assumed to depend on the current state of deformation and the magnetic induction vector in the deformed configuration $\boldsymbol{B}$. There are two additional magnetic vectors and in the deformed configuration they are, $\boldsymbol{H}$, the magnetic field vector and $\boldsymbol{M}$ the magnetisation vector. The three vectors are related by

$$\boldsymbol{B} = \mu_0 \left(\boldsymbol{H} + \boldsymbol{M}\right),$$

where μ_0 is a constant, the magnetic permeability of vacuum.

(i) The additional rate of work due to the electromagnetic effects is $-\boldsymbol{M}\boldsymbol{\cdot}\dot{\boldsymbol{B}}$, where the dot indicates a material derivative. Use the first and second laws of thermodynamics to demonstrate that

$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} + \mathsf{T} : \mathsf{D} - \boldsymbol{M}\boldsymbol{\cdot}\dot{\boldsymbol{B}} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta \geq 0,$$

where $\Psi = \Phi - \eta\Theta$ is the Helmholtz free energy function and the dot indicates the material derivative.

(ii) Assuming a constant and uniform temperature and that the material is incompressible show that in the Lagrangian description

$$-\frac{D\omega}{Dt} + \hat{s}^{ij}\dot{\gamma}_{ij} + h^i\dot{b}_i \geq 0,$$

where $\boldsymbol{h}$ and $\boldsymbol{b}$ are the magnetic field and magnetic induction vectors in the undeformed configuration respectively,

$$\omega = \rho\Psi + \frac{1}{2\mu_0}\boldsymbol{B} \cdot \boldsymbol{B},$$

and $\hat{s}^{ij}$ is a modified second Piola–Kirchhoff stress tensor that is to be found. Note that $\boldsymbol{B} = \mathsf{F}\boldsymbol{b}$ and $\boldsymbol{H} = \mathsf{F}^{-T}\boldsymbol{h}$, where F is the deformation gradient tensor.

(iii) Assuming that ω is a function only of γ_{ij} and b_i and that mechanical and magnetic processes are independent show that

$$\hat{s}^{ij} = \frac{\partial\omega}{\partial\gamma_{ij}} \quad \text{and} \quad h^i = \frac{\partial\omega}{\partial b_i}.$$

[18 marks]

B7. In a Newtonian fluid the Cauchy stress is given by

$$\mathsf{T} = -P\mathsf{I} + \lambda(\boldsymbol{\nabla}_{\!R}\boldsymbol{\cdot}\boldsymbol{V})\,\mathsf{I} + 2\mu\mathsf{D},$$

where $D_{IJ} = \frac{1}{2}\left(V_{I,J} + V_{J,I}\right)$ in the usual notation.

(i) Show that the thermodynamic pressure P is not equal to the mechanical pressure $P_m = -\mathrm{tr}(\mathsf{T})/3$ unless either (i) the fluid is incompressible or (ii) $\lambda + 2\mu/3 = 0$.

(ii) Show that the Eulerian rate of deformation tensor D can be decomposed into the form

$$\mathsf{D} = \widetilde{\mathsf{D}} + \frac{1}{3}\left(\boldsymbol{\nabla}_{\!R}\boldsymbol{\cdot}\boldsymbol{V}\right)\mathsf{I},$$

where $\widetilde{\mathsf{D}}$, the deviatoric part of the tensor, is such that $\mathrm{trace}(\widetilde{\mathsf{D}}) = 0$.

(iii) Thermodynamic constraints mean that the Helmholtz free energy is a function only of density and temperature, $\Psi(\rho, \Theta)$, and

$$\frac{\partial \Psi}{\partial \Theta} = -\eta \quad \text{and} \quad \frac{\partial \Psi}{\partial \rho} = \frac{P}{\rho^2}.$$

If the heat flux is given by $\boldsymbol{Q} = -\kappa \boldsymbol{\nabla}_{\!R}\Theta$, show that the Clausis–Duhem inequality is satisfied by every admissible thermomechanical process if

$$\mu \geq 0, \quad \lambda + \frac{2}{3}\mu \geq 0, \quad \kappa \geq 0.$$

[18 marks]

FORMULA SHEET

- For a general (Lagrangian) coordinate system ξ^i:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \xi^i}, \quad \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}^j = \delta_i^j, \quad g_{ij} = \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}_j, \quad g = \det(g_{ij}).$$

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- For a scalar field $f(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$

$$\boldsymbol{\nabla} f = \boldsymbol{g}^i \frac{\partial f}{\partial \xi^i}, \quad \operatorname{div} \boldsymbol{u} = \frac{1}{\sqrt{g}} \frac{\partial \left(u^i \sqrt{g}\right)}{\partial \xi^i}, \quad \operatorname{curl} \boldsymbol{u} = \epsilon^{ijk} u_j|_i \boldsymbol{g}_k.$$

- The material derivative in general coordinates is

$$\frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + V^j U^i||_j,$$

where $\boldsymbol{V}$ is the velocity of the continuum and

$$U^i||_j = U^{i,j} + \Gamma^i_{jk} U^k,$$

where Γ^i_{jk} are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

- Cauchy's equation in the usual notation in components in general coordinates ξ^i is

$$T^{ji}||_j + \rho F^i = \rho \ddot{U}^i = \rho \frac{DV^i}{Dt}, \quad \text{where} \quad T^{ji}||_j = T^{ji}_{,j} + \Gamma^j_{jr} T^{ri} + \Gamma^i_{jr} T^{jr}.$$

- The material derivative of the determinant of the deformation gradient tensor is

$$\frac{DJ}{Dt} = J \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V}.$$

- The Reynolds Transport theorem states that

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_t} \phi \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \left(\frac{D\phi}{Dt} + \phi \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} \right) \mathrm{d}\mathcal{V}_t,$$

where Ω_t is a material volume, ϕ is a scalar field and $\boldsymbol{V}$ is the velocity of the continuum.

- For a Cartesian line element $\mathrm{d}X_I$ in the deformed configuration

$$\frac{D\mathrm{d}X_I}{Dt} = V_{I,K} \mathrm{d}X_K,$$

where V_I is the I-th Cartesian component of the velocity.

- Nanson's relation states that
$$\mathrm{d}A_{\bar{i}} = J\frac{\partial \xi^j}{\partial \chi^{\bar{i}}}\mathrm{d}a_j,$$
where ξ^j are the Lagrangian coordinates, $\chi^{\bar{i}}$ are the Eulerian coordinates, J is the determinant of the deformation gradient tensor, $\mathrm{d}\boldsymbol{A}$ is an area element in the deformed configuration and $\mathrm{d}\boldsymbol{a}$ is an area element in the undeformed configuration.

- The Green–Lagrange strain tensor is defined by
$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right).$$

- The strain invariants are defined by
$$I_1 = g^{ij}G_{ji}, \quad I_2 = \frac{1}{2}\left(I_1^2 - g^{ir}g^{js}G_{ij}G_{rs}\right), \quad I_3 = G/g,$$
where $g = det(g_{ij})$ and $G = det(G_{ij})$

- A hyperelastic material is described by a strain energy function $\mathcal{W}(I_1, I_2, I_3)$ such that
$$T^{ij} = PG^{ij} + Ag^{ij} + BB^{ij},$$
where
$$A = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_1}, \quad B = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_2}, \quad P = 2\sqrt{I_3}\frac{\partial \mathcal{W}}{\partial I_3},$$
$$\text{and} \quad B^{ij} = \left[I_1 g^{ij} - g^{ir}g^{js}G_{rs}\right].$$

- The physical components of the stress tensor are given by $\sigma_{(ij)} = T^{ij}\sqrt{G_{jj}/G^{ii}}$ (no summation).

- The body stress tensor T^{ij} and second Piola–Kirchhoff stress tensor s^{ij} are related by the expression $JT^{ij} = s^{ij}$.

- The first law of thermodynamics can be written as
$$\rho\frac{D\Phi}{Dt} = \mathsf{T} : \mathsf{D} + \rho B - \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{Q} + \mathcal{W}_e,$$
where $\mathcal{W}_e$ is any additional non-thermomechanical rates of work.

- The second law of thermodynamics for continuum mechanics can be written as
$$\rho\dot{\eta} \geq -\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\left(\frac{\boldsymbol{Q}}{\Theta}\right) + \rho\frac{B}{\Theta}.$$

- The Clausius–Duhem inequality is
$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta + \mathsf{T} : \mathsf{D} \geq 0,$$
where $\Psi = \Phi - \eta\Theta$.

- The most general transformation of position and time between observers in Euclidean space is
$$\boldsymbol{R}^*(t^*) = \mathsf{Q}(t)\boldsymbol{R}(t) + \boldsymbol{C}(t), \quad t^* = t - a,$$
where Q is an orthogonal matrix, $\boldsymbol{C}$ is a translation vector and a is a constant time shift.

END OF EXAMINATION PAPER