Three hours

THE UNIVERSITY OF MANCHESTER

CONTINUUM MECHANICS

24 January 2017

09:45 – 12:45

Answer **ALL THREE** questions in section A (21 marks in total).
Answer **THREE** of the **FOUR** questions in section B (54 marks in total). If more than THREE questions from Section B are attempted, then credit will be given for the best THREE answers.

University approved calculators may be used.

SECTION A

A1. Newton's law for a particle can be written as $\boldsymbol{F} = m\boldsymbol{A}$, where $\boldsymbol{F}$ is the resultant force, m is the mass of the particle and $\boldsymbol{A}$ is the acceleration.

(i) Define what is meant by an objective quantity.

(ii) By direct differentiation of the position explain why Newton's second law for a particle is **not** objective.

[5 marks]

A2. Small particles are advected by a moving continuum, where $N(\boldsymbol{R}, t)$ denotes the number of particles per unit volume, $\boldsymbol{R}$ is the Eulerian position and t is time.

(i) If the total number of particles within a material region Ω_t is conserved, show that this condition is described by the equation

$$\frac{\partial N}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot (N\boldsymbol{V}) = 0,$$

where $\boldsymbol{V}$ is the velocity of the continuum.

(ii) Derive the modified conservation equation when there is a source of particles, $S(\boldsymbol{R}, t)$, within the continuum. Be sure to specify the units of S carefully.

[6 marks]

A3. Consider a sphere of solid material, initially of unit radius, that is first uniformly dilated into a sphere of radius r_1 and subsequently to a sphere of radius r_2. You may assume a linear stretch: the deformed radius $R = \lambda r$, where λ is a constant to be determined for each deformation. A standard spherical polar coordinate system $(\xi^1, \xi^2, \xi^3) \equiv (r, \theta, \phi)$ is defined with origin at the centre of the undeformed sphere, such that the undeformed position is given by

$$\boldsymbol{r} = r \sin\theta \cos\phi \boldsymbol{e}_X + r \sin\theta \sin\phi \boldsymbol{e}_Y + r \cos\theta \boldsymbol{e}_Z,$$

where $\boldsymbol{e}_X$, $\boldsymbol{e}_Y$ and $\boldsymbol{e}_Z$ are unit vectors in a global Cartesian coordinate system.

(i) Find the Green–Lagrange strain tensor associated with the first deformation, $\gamma_{ij}^{(1)}$.

(ii) Find the Green–Lagrange strain tensor associated with the subsequent (second) deformation, $\gamma_{ij}^{(2)}$, assuming that the reference configuration is the state after the first deformation.

(iii) Find the strain tensor associated with the total deformation from initial configuration to final configuration, γ_{ij}.

(iv) Find a relationship between the three strain tensors, γ_{ij}, $\gamma_{ij}^{(1)}$ and $\gamma_{ij}^{(2)}$.

[10 marks]

SECTION B

B4. An incompressible material has the constitutive relationship

$$\mathsf{T} = -P\mathsf{I} + \eta_s \mathsf{D} + \widehat{\mathsf{T}},$$

where P is the pressure, η_s is a constant and $\widehat{\mathsf{T}}$ satisfies

$$\widehat{\mathsf{T}} + \lambda_1 \widehat{\mathsf{T}}^{\nabla} + \alpha \frac{\lambda_1}{\eta_p} \widehat{\mathsf{T}}^2 = \eta_p \mathsf{D},$$

where η_p, λ_1 and α are constants. The upper-convected derivative is defined by

$$\mathsf{A}^{\nabla} = \frac{D\mathsf{A}}{Dt} - \mathsf{L}\mathsf{A} - \mathsf{A}\mathsf{L}^T,$$

where $\mathsf{L} = \boldsymbol{\nabla}_{\boldsymbol{R}} \otimes \boldsymbol{V}$ is the Eulerian velocity gradient tensor and D is the symmetric part of L.

(i) Confirm that the constitutive relationships are objective.

You may assume that the Cauchy stress T is objective; that the deformation gradient tensor, F, transforms as $\mathsf{F}^* = \mathsf{Q}\mathsf{F}$, where Q is an orthogonal matrix that expresses the relative rotation between observers; and that $\mathsf{L} = \dot{\mathsf{F}}\mathsf{F}^{-1}$.

A steady two-dimensional extensional flow is given by $\boldsymbol{V} = (\epsilon X, -\epsilon Y)$, where ϵ is a constant rate of extension and X and Y are Cartesian coordinates.

(ii) Assuming now that $\widehat{\mathsf{T}}$ is constant in time and space, find the Cartesian components of $\widehat{\mathsf{T}}$ in this flow. Comment on any choice of roots. (You may assume that $1 + (\alpha\lambda_1/\eta_p)(\widehat{T}_{XX} + \widehat{T}_{YY}) \neq 0$.)

(iii) Comment on the validity of the constitutive model by considering what happens to the stress component $\widehat{\mathsf{T}}_{XX}$ as $\epsilon \to 1/(2\lambda_1)$ in the cases when $\alpha \neq 0$ and $\alpha = 0$.

[18 marks]

B5. Consider a hyperelastic solid that also undergoes isotropic growth.

(i) If M is the rate of change of mass per unit deformed volume show by using the integral mass balance that the Eulerian form of the statement of conservation of mass is

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{V} = M.$$

(ii) Show that

$$\frac{D}{Dt}\int_{\Omega_t} \rho\phi \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \left(\rho \frac{D\phi}{Dt} + M\phi\right) \mathrm{d}\mathcal{V}_t,$$

where ϕ is a scalar field and Ω_t is a material volume. Hence show that the balance of linear momentum equation is unchanged in the presence of growth, under the assumption that the additional momentum due to growth is $M\boldsymbol{V}$ per unit deformed volume. Interpret this assumption.

(iii) Growth can be modelled by assuming that the undeformed metric tensor (in a Cartesian coordinate system) is

$$g_{IJ} = \lambda_g^2(t)\, \delta_{IJ},$$

where $\lambda_g(t)$ is a function only of time that is related to the growth. The growing material is subject to a uniaxial extension such that

$$X_1 = \lambda_1 x_1, \quad X_2 = \lambda_2 x_2, \quad X_3 = \lambda_2 x_3.$$

Find G_{ij} and hence find the values of λ_1 and λ_2 such that the Green–Lagrange strain tensor is zero.

B6. For an ideal gas the internal energy, Φ, Cauchy stress, T, and entropy, η, depend only on the density, ρ, and temperature, Θ:

$$\Phi(\rho,\Theta), \quad \mathsf{T}(\rho,\Theta), \quad \eta(\rho,\Theta).$$

(i) Use the fact that these quantities must remain indifferent under a change in Eulerian observer given by $\boldsymbol{R}^* = \mathsf{Q}(t)\boldsymbol{R}$ to deduce that

$$\mathsf{T} = -P(\rho,\Theta)\mathsf{I}, \tag{1}$$

where P is called the thermodynamic pressure and I is the identity tensor.

(ii) Use the Clausius–Duhem inequality in the absence of heat flux, together with conservation of mass to show that

$$P = \rho^2\frac{\partial \Phi}{\partial \rho} - \rho^2\Theta\frac{\partial \eta}{\partial \rho}, \qquad \frac{\partial \Phi}{\partial \Theta} = \Theta\frac{\partial \eta}{\partial \Theta}.$$

(iii) Show that the ideal gas constitutive laws

$$\Phi = c_v\Theta, \qquad P = \rho R\Theta,$$

satisfy the equations deduced in (ii) provided that η has a particular form, which is to be found. Here c_v is a constant, the specific heat of the gas at constant volume, and R is the ideal gas constant.

[18 marks]

B7. An incompressible, hyperelastic solid cylinder has undeformed radius 1 and undeformed length 2. The cylinder undergoes a deformation such that it remains cylindrical with no twist and the deformed position is given by $R(r)$, $\Theta = \theta$ and $Z(z)$. The strain energy function of the solid material is given by

$$\mathcal{W} = C_1(I_1 - 3)^n + C_2(I_2 - 3)^m,$$

where m, n, C_1 and C_2 are constants.

(i) Write down the deformed position in components in a global Cartesian coordinate system as a function of the undeformed cylindrical polar coordinates (r, θ, z).

(ii) Hence, compute the three strain invariants corresponding to this deformation.

(iii) Use the incompressibility constraint to deduce the most general forms of R and Z and find corresponding values of the strain invariants.

(iv) Compute the stress tensor T^{ij} in the deformed solid and confirm that Cauchy's equations are satisfied in the absence of body forces and accelerations.

Hint: In cylindrical polar coordinates $(\xi^1, \xi^2, \xi^3) = (r, \theta, z)$ the only non-zero Christoffel symbols are given by

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \quad \text{and} \quad \Gamma^1_{22} = -r.$$

[18 marks]

FORMULA SHEET

- For a general (Lagrangian) coordinate system ξ^i:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \xi^i}, \quad \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}^j = \delta_i^j, \quad g_{ij} = \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}_j, \quad g = \det(g_{ij}).$$

$$\boldsymbol{G}_i = \frac{\partial \boldsymbol{R}}{\partial \xi^i}, \quad \boldsymbol{G}_i \boldsymbol{\cdot} \boldsymbol{G}^j = \delta_i^j, \quad G_{ij} = \boldsymbol{G}_i \boldsymbol{\cdot} \boldsymbol{G}_j, \quad G = \det(G_{ij}).$$

- For a scalar field $f(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$

$$\boldsymbol{\nabla} f = \boldsymbol{g}^i \frac{\partial f}{\partial \xi^i}, \quad \operatorname{div} \boldsymbol{u} = \frac{1}{\sqrt{g}} \frac{\partial \left(u^i \sqrt{g}\right)}{\partial \xi^i}, \quad \operatorname{curl} \boldsymbol{u} = \epsilon^{ijk} u_j|_i \boldsymbol{g}_k.$$

- The material derivative in general coordinates is

$$\frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + V^j U^i||_j,$$

where $\boldsymbol{V}$ is the velocity of the continuum.

- Cauchy's equation in the usual notation in components in general coordinates ξ^i is

$$T^{ji}||_j + \rho F^i = \rho \ddot{U}^i = \rho \frac{DV^i}{Dt}, \quad \text{where} \quad T^{ji}||_j = T^{ji}_{,j} + \Gamma^j_{jr} T^{ri} + \Gamma^i_{jr} T^{jr},$$

and Γ^i_{jk} are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

- The material derivative of the determinant of the deformation gradient tensor is

$$\frac{DJ}{Dt} = J \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V}.$$

- The Reynolds Transport theorem states that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \phi \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \left(\frac{D\phi}{Dt} + \phi \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} \right) \mathrm{d}\mathcal{V}_t,$$

where Ω_t is a material volume, ϕ is a scalar field and $\boldsymbol{V}$ is the velocity of the continuum.

- For a Cartesian line element $\mathrm{d}X_I$ in the deformed configuration

$$\frac{D \mathrm{d}X_I}{Dt} = V_{I,K} \mathrm{d}X_K,$$

where V_I is the I-th Cartesian component of the velocity.

- Nanson's relation states that

$$\mathrm{d}A_{\bar{i}} = J \frac{\partial \xi^j}{\partial \chi^{\bar{i}}} \mathrm{d}a_j,$$

where ξ^j are the Lagrangian coordinates, $\chi^{\bar{i}}$ are the Eulerian coordinates, J is the determinant of the deformation gradient tensor, $\mathrm{d}\boldsymbol{A}$ is an area element in the deformed configuration and $\mathrm{d}\boldsymbol{a}$ is an area element in the undeformed configuration.

- The Green–Lagrange strain tensor is defined by

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right).$$

- The strain invariants are defined by

$$I_1 = g^{ij}G_{ji}, \quad I_2 = \frac{1}{2}\left(I_1^2 - g^{ir}g^{js}G_{ij}G_{rs}\right), \quad I_3 = G/g,$$

  where $g = det(g_{ij})$ and $G = det(G_{ij})$

- A hyperelastic material is described by a strain energy function $\mathcal{W}(I_1, I_2, I_3)$ such that

$$T^{ij} = PG^{ij} + Ag^{ij} + BB^{ij},$$

  where

$$A = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_1}, \quad B = \frac{2}{\sqrt{I_3}}\frac{\partial \mathcal{W}}{\partial I_2}, \quad P = 2\sqrt{I_3}\frac{\partial \mathcal{W}}{\partial I_3},$$
$$\text{and} \quad B^{ij} = \left[I_1 g^{ij} - g^{ir}g^{js}G_{rs}\right].$$

- The physical components of the stress tensor are given by $\sigma_{(ij)} = T^{ij}\sqrt{G_{jj}/G^{ii}}$ (no summation).

- The body stress tensor T^{ij} and second Piola–Kirchhoff stress tensor s^{ij} are related by the expression $JT^{ij} = s^{ij}$.

- The Clausius–Duhem inequality is

$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta + \mathsf{T} : \mathsf{D} \geq 0,$$

  where $\Psi = \Phi - \eta\Theta$.

- The most general transformation of position and time between observers in Euclidean space is

$$\boldsymbol{R}^*(t^*) = \mathsf{Q}(t)\boldsymbol{R}(t) + \boldsymbol{C}(t), \quad t^* = t - a,$$

  where Q is an orthogonal matrix, $\boldsymbol{C}$ is a translation vector and a is a constant time shift.

END OF EXAMINATION PAPER