The majority of the students attempted all questions in section A and attempts at section B were evenly distributed over the questions. Most students got nearly full marks on questions A1 and A2. Question A3 proved more of a challenge than I expected, with only a few students obtaining the majority of the marks. The section B questions were generally balanced, with most students getting around about half marks on each question. The overall average mark for the course was approximately 60%. One common theme was difficulty in interpreting what the mathematics actually means (particularly question A3 and B4). Another thing that made life more difficult for some was not making effective use of the formulæ given at the end of the examination.

A1 This question tested basic understanding of the fundamental differences in constitutive behaviour of solids and fluids which was discussed (a lot) in the course.

(i) The key concepts that should have been mentioned were that (a) a solid has a preferred rest state whereas a fluid doesn't; and (b) strains induce stresses in solids, but not in (pure) fluids where it is instead the rate of strain that matters. A common error was believing the incompressibility was related to fluid or solid behaviour. It isn't; you can have compressible fluids (air) and solids. In fact, incompressibility is a convenient mathematical idealisation ... anything will compress, if you try hard enough.

(ii) Most people got this question correct. The answers were (a) fluid-like (dependence on rate of strain), (b) solid (dependence on strain), (c) mixture (dependence on strain and rate of strain).

A2 This question tested understanding of the concept of conservation of mass and how to derive the governing equation in PDE form from an integral definition. It was well answered by the vast majority of students.

A3 This question tested understanding of deformation, deformed and undeformed metrics and what it means for a deformation to be physically admissible. I was surprised at how poorly this question was answered, which may reflect a general lack of confidence with polar coordinates. Only a handful of people were able to describe the deformed configuration and the vast majority made very heavy weather of the calculations, which were often based on an inappropriate assumption that the deformed radius was independent of θ.

(i) Direct substitution (which is an obvious thing to do, but very few did) gives the deformed position as

$$\boldsymbol{R} = \begin{pmatrix} \frac{rL}{a} \\ \frac{rL}{a}\tan\theta \\ \frac{Hz}{h} \end{pmatrix}, \quad \theta \in [0, \pi/4], \qquad \begin{pmatrix} \frac{rL}{a}\cot\theta \\ \frac{rL}{a} \\ \frac{Hz}{h} \end{pmatrix}, \quad \theta \in [\pi/4, \pi/2].$$

This is the deformed position in the global Cartesian coordinate system, so when $\theta \leq \pi/4$, then $X = rL/a$, a constant and Y varies from 0 to rL/a. When $\theta \geq \pi/4$, $Y = rL/a$, a constant and X varies from rL/a to 0. Thus, the boundary is deformed into a square of side length L.

(ii) Direct differentiation of $\boldsymbol{R}$ found above gives the deformed metric tensor and the deformation is always physically admissible because $\det G_{ij}/\det g_{ij} > 0$.

B4 The initial parts of this question were answered well, but the later parts were more challenging, despite the fact that this question is similar to Example Sheet V, q 1.

(i) Most people got the majority of the marks here, but an important assumption is that the coordinate planes remain aligned and that there is no twist between planes.

(ii) The majority correctly computed the invariants. One common error was to replace the metric tensor by its square-root.

(iii) Many people correctly used the constitutive law to find the components of the body stress tensor. A number of people forgot that the physical tension T acts in the axial (x_1 direction) and must balance the physical component of stress, which is $\lambda^2 T^{11}$. The components of stress in the transverse directions T^{22} and T^{33} are both zero, which can be used to determine μ as a function λ and then the axial component gives the expression for T.

(iv) Here, the most common error was forgetting that when a material is incompressible P is an independent variable. P can be found from the condition that the transverse stress is zero and then T again follows from the axial stress balance.

B5 This question was answered well by most that attempted it.

(i) All students correctly summed the individual mass conservation equations to obtain the required result for the mixture. Most also obtained the momentum equation, but establishing the intermediate relationship

$$\rho_a \frac{D\boldsymbol{V}_a}{Dt} = \frac{\partial}{\partial t}\left(\rho_a \boldsymbol{V}_a\right) + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \left(\rho_a \boldsymbol{V}_a \otimes \boldsymbol{V}_a\right),$$

proved to be more difficult. Working in index notation (in Cartesians, but you can work in general coordinates if you prefer):

$$\rho_a \frac{DV_{a\,I}}{Dt} = \frac{D}{Dt}\left(\rho_a V_{a\,I}\right) - V_{a\,I}\frac{D\rho_a}{Dt},$$

and then using the conservation of mass equation gives

$$\rho_a \frac{DV_{a\,I}}{Dt} = \frac{D}{Dt}\left(\rho_a V_{a\,I}\right) + \rho_a V_{a\,I}\left(V_{a\,J}\right)_{,J} = \frac{\partial}{\partial t}\left(\rho_a V_{a\,I}\right) + V_{a\,J}\left(\rho_a V_{a\,I}\right)_{,J} + \rho_a V_{a\,I} V_{a\,J,J},$$

and so

$$\rho_a \frac{DV_{a\,I}}{Dt} = \frac{\partial}{\partial t}\left(\rho_a V_{a\,I}\right) + \left(\rho_a V_{a\,I} V_{a\,J}\right)_{,J},$$

the desired result.

(ii) This correctly answered by most students and follows from direct differentiation of the heat flux term, use of the energy equation to replace $\boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{Q}$ and introduction of the Helmholtz free energy $\Psi = \Phi - \eta\Theta$.

(iii) This follows the standard line in the lecture notes, but with the additional variables C_a. Expanding out the partial derivative of Ψ using the chain rule and using conservation of mass to replace $D\rho/Dt$ gives

$$\left[\frac{\partial \Psi}{\partial \rho}\rho^2 \mathsf{I} + \mathsf{T}\right] : \mathsf{D} - \left[\rho\eta + \rho\frac{\partial \Psi}{\partial \Theta}\right]\dot{\Theta} - \sum_a \rho \frac{\partial \Psi}{\partial C_a}\dot{C}_a - \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{J} - \frac{1}{\Theta}\boldsymbol{Q} \cdot \boldsymbol{\nabla}_{\boldsymbol{R}}\Theta \geq 0.$$

The first two terms we have seen before in other examples and choosing isothermal deformations or deformationless temperature variations allows us to deduce the equations for η and T. The remaining terms lead to the expression

$$\sum_a \rho \frac{\partial \Psi}{\partial C_a}\dot{C}_a + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \boldsymbol{J} + \frac{1}{\Theta}\boldsymbol{Q} \cdot \boldsymbol{\nabla}_{\boldsymbol{R}}\Theta \leq 0,$$

from which the desired result follows if

$$\mu_a = -\rho \frac{\partial \Psi}{\partial C_a}.$$

B6 The main problem with this question was taking the infinitesimal limit in (ii), which, of course, makes (iii) rather difficult. We had seen the technique in lectures and example sheets, but, of course, one needs to work with a technique to become confident with it.

(i) This was a matter of using the formulæ and differentiating the strain energy function with respect to I_1, I_2 and I_3. Most students got full marks on this part.

(ii) These derivations had been seen in the lecture notes and used in the example sheets. However, using the given results directly in the expression for the stress one obtains

$$s^{ij} \approx \tau^{ij} \approx C_2 \log(1 + 2e^k_k)^{1/2} g^{ij} + 2C_1 e^{ij} \approx C_2 e^k_k g^{ij} + 2C_1 e^{ij},$$

to leading order.

(iii) Direct comparison of the above expression with the linear constitutive law gives $C_2 = \lambda$ and $C_1 = \mu$.

B7 This question was generally well answered.

(i) This was standard bookwork combined with general principles of objectivity: i.e. sum of objective quantities remains objective.

(ii) Everybody correctly computed D and T_1. The most common error in computing T_2 was going for the momentum (Cauchy's) equation rather than the given equation. The stress is constant in time and space and there are no body forces so Cauchy's equation is trivially satisfied. Substituting the given form for D and L into the governing equation for T_2 and using the fact that the stress tensor is symmetric leads to three equations for $T_{2\,XX}$, $T_{2\,YY}$ $T_{2\,XY}$. The off-diagonal terms $T_{2\,XY} = 0$ and equations for the other two components then decouple leading to two different quadratic equations. The positive root of each quadratic must be taken so that the stress remains well-defined in the limit $\alpha \to 0$.