Three hours

THE UNIVERSITY OF MANCHESTER

CONTINUUM MECHANICS

23 January 2015

14:00 – 17:00

Answer **ALL FOUR** questions in section A (21 marks in total).
Answer **THREE** of the **FOUR** questions in section B (54 marks in total). If more than THREE questions from Section B are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

A1.

(i) If a general coordinate system is orthogonal explain why the covariant metric tensor takes the form

$$g_{ij} = \begin{pmatrix} (h_1)^2 & 0 & 0 \\ 0 & (h_2)^2 & 0 \\ 0 & 0 & (h_3)^2 \end{pmatrix}$$

and find explicit expressions for the entries h_i in terms of the covariant base vectors $\boldsymbol{g}_i$.

(ii) Find an expression for the Laplacian of a scalar field $\boldsymbol{\nabla}^2 u = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u$ in terms of the quantities h_i and partial derivatives with respect to ξ^i.

[6 marks]

A2. Explain why a constitutive law of the form

$$\frac{D\mathsf{T}}{Dt} = \mathbf{A}\mathsf{D}$$

is not objective. Here, T is the Cauchy stress tensor; $\mathsf{D} = (\mathsf{L} + \mathsf{L}^T)/2$ is the Eulerian rate of deformation tensor, $\mathsf{L} = \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{V}$; and $\mathbf{A}$ is a constant fourth-rank tensor. Give an example of what could be done to make the law objective.

[5 marks]

A3. Two different miscible fluids have densities ρ_1 and ρ_2 and respective velocities $\boldsymbol{V}_1(\boldsymbol{R}, t)$ and $\boldsymbol{V}_2(\boldsymbol{R}, t)$. The respective production rates of each fluid per unit volume are $S_1(\boldsymbol{R}, t)$ and $S_2(\boldsymbol{R}, t)$.

(i) By considering the conservation of mass of each fluid within a control volume show that

$$\frac{D\rho_1}{Dt} + \rho_1 \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V}_1 = S_1, \quad \frac{D\rho_2}{Dt} + \rho_2 \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V}_2 = S_2.$$

(ii) Hence, show that conservation of total mass is expressed by

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} = S_1 + S_2,$$

where $\rho = \rho_1 + \rho_2$ and $\boldsymbol{V}$ is to be found.

(iii) If the total mass must be conserved state the constraint on S_1 and S_2.

[6 marks]

A4. A continuum is deformed from an initial configuration with position vector $\boldsymbol{r}$ to a new configuration described by position vector $\boldsymbol{R}$. The initial position is described by a (Lagrangian) coordinate system ξ^i, the new position by an (Eulerian) coordinate system $\chi^{\bar{j}}$.

The force acting on an infinitesimal surface of a continuum is given by

$$\boldsymbol{F} = T^{\bar{i}\bar{j}} \boldsymbol{G}_{\bar{j}} \mathrm{d}A_{\bar{i}},$$

where T is the Cauchy stress tensor; $\mathrm{d}\boldsymbol{A}$ is the deformed surface area; and $\boldsymbol{G}_{\bar{j}} = \frac{\partial \boldsymbol{R}}{\partial \chi^{\bar{j}}}$. The second Piola–Kirchhoff stress tensor, s, is defined by the relationship

$$\boldsymbol{F} = s^{ij} \boldsymbol{G}_j \mathrm{d}a_i,$$

where $\mathrm{d}\boldsymbol{a}$ is the undeformed surface area; and $\boldsymbol{G}_j = \frac{\partial \boldsymbol{R}}{\partial \xi^j}$.

Find the relationship between s and T.

[4 marks]

SECTION B

B5. A spherical ball of hyperelastic material is deformed such that material originally at radius r is moved to the radius $R(r)$, but the ball maintains its spherical shape.

(i) By computing the eigenvalues of the mixed metric tensor $g^{ij}G_{jk}$, or otherwise, show that the stretch in the radial direction, λ_r, is given by $\frac{\mathrm{d}R}{\mathrm{d}r}$, whereas the stretch in the directions tangential to the sphere's surface is given by $\lambda_t = R/r$.

(ii) Show that the three strain invariants are given by

$$I_1 = \lambda_r^2 + 2\lambda_t^2, \quad I_2 = \lambda_t^4 + 2\lambda_r^2\lambda_t^2, \quad I_3 = \lambda_r^2\lambda_t^4.$$

(iii) By using the properties of I_3 show that $R = r$ for an incompressible material.

You should assume that spherical coordinates are defined by $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = \phi$, where the undeformed position is given by $\boldsymbol{r} = r\sin\theta\cos\phi\,\boldsymbol{e}_x + r\sin\theta\sin\phi\,\boldsymbol{e}_y + r\cos\theta\,\boldsymbol{e}_z$, where $\boldsymbol{e}_I$ are global Cartesian base vectors.

[18 marks]

B6. A hyperelastic material consists of a family of fibres embedded in a solid matrix that undergoes a general deformation from an undeformed configuration described by the position vector $\boldsymbol{r}$ to a current position described by $\boldsymbol{R}$. The direction of the fibres in the undeformed configuration is represented by the unit direction vectors, $\boldsymbol{a}(\boldsymbol{r}, t)$.

(i) The second Piola–Kirchhoff stress tensor can be derived from a strain energy $\mathcal{W}$ by the expression

$$s^{ij} = \frac{\partial \mathcal{W}}{\partial \gamma_{ij}},$$

where γ_{ij} is the Green–Lagrange strain tensor. By converting to Cartesian coordinates (or otherwise) show that

$$\mathsf{s} = 2\frac{\partial \mathcal{W}}{\partial \mathsf{c}},$$

where $\mathsf{c} = \mathsf{F}^T\mathsf{F}$ and $\mathsf{F} = \boldsymbol{\nabla}_{\boldsymbol{r}}\boldsymbol{R}$ is the deformation gradient tensor.

(ii) Show that the unit direction vectors become $\boldsymbol{A} = \mathsf{F}\boldsymbol{a}$, in the deformed configuration.

(iii) The strain energy can be written in the form $\mathcal{W}(\mathsf{c}, \boldsymbol{a} \otimes \boldsymbol{a})$ and there are now five possible invariants

$$I_1 = \text{trace}(\mathsf{c}), \quad I_2 = \frac{1}{2}\left\{[\text{trace}(\mathsf{c})]^2 - \text{trace}(\mathsf{c}^2)\right\}, \quad I_3 = \det(\mathsf{c}), \quad I_4 = \boldsymbol{a}\boldsymbol{\cdot}\mathsf{c}\boldsymbol{a}, \quad I_5 = \boldsymbol{a}\boldsymbol{\cdot}\mathsf{c}^2\boldsymbol{a}.$$

Hence, find an expression for the second Piola–Kirchhoff stress tensor of the form

$$\mathsf{s} = \sum_{\alpha=1}^{5} \mathsf{D}_\alpha \frac{\partial W}{\partial I_\alpha},$$

where D_α are all second-rank tensors that are to be found.
You may assume the result $\partial I_3/\partial \mathsf{c} = I_3 \mathsf{c}^{-1}$.

(iv) If the strain energy function is given by

$$\mathcal{W} = \frac{1}{2}(I_1 - 3) - \frac{1}{2}p\,(I_3 - 1) - \frac{1}{2}q\,(I_4 - 1),$$

where p and q are constants, find the form of the corresponding Cauchy stress, T.
You may use the relationship $\mathsf{s} = J\mathsf{F}^{-1}\mathsf{T}\mathsf{F}^{-T}$ without proof.

[18 marks]

B7. The Rivlin–Ericksen tensor of order m, $\mathsf{A}^{(m)}$, is defined by

$$\frac{D^m}{Dt^m}|\mathrm{d}\boldsymbol{R}|^2 = \mathsf{A}^{(m)}_{\bar{i}\,\bar{j}}\,\mathrm{d}\chi^{\bar{i}}\,\mathrm{d}\chi^{\bar{j}},$$

where $\chi^{\bar{i}}$ are general Eulerian coordinates.

(i) Show that $\mathsf{A}^{(1)} = \mathsf{L} + \mathsf{L}^T$, where $\mathsf{L} = \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{V}$ is the velocity gradient tensor.

(ii) Prove the recurrence relationship

$$\mathsf{A}^{(m+1)} = \frac{D\mathsf{A}^{(m)}}{Dt} + \mathsf{L}^T\mathsf{A}^{(m)} + \mathsf{A}^{(m)}\mathsf{L}.$$

(iii) A third-order incompressible fluid has Cauchy stress given by

$$\mathsf{T} = \left\{a_1 + a_2\mathrm{trace}\left[\left(\mathsf{A}^{(1)}\right)^2\right]\right\}\mathsf{A}^{(1)} + a_3\mathsf{A}^{(2)} + a_4\left(\mathsf{A}^{(1)}\right)^2 + a_5\mathsf{A}^{(3)} + a_6\left(\mathsf{A}^{(1)}\mathsf{A}^{(2)} + \mathsf{A}^{(2)}\mathsf{A}^{(1)}\right),$$

where the quantities a_i are constants.

(a) A simple shear flow in Cartesian coordinates is given by $V_1 = \gamma X_2$, $V_2 = V_3 = 0$, where γ is a constant shear rate. Find an explicit expression for the shear stress, T_{12}, of the third-order fluid and hence find an expression for the effective viscosity $\mu(\gamma)$ such that

$$T_{12} = \mu(\gamma)\gamma.$$

(b) If the fluid is shear-thinning (μ decreases with increasing γ) show that if the shear stress is to increase monotonically with shear rate that there is a maximum allowable shear rate. Hence comment on the validity of the model.

[18 marks]

B8.

(i) A perfect thermoelastic material is modelled using the constitutive assumptions (in the usual notation)

$$\Psi = \Psi(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta), \quad \mathsf{T} = \mathsf{T}(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta), \quad \eta = \eta(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta), \quad \boldsymbol{Q} = \boldsymbol{Q}(\mathsf{F}, \Theta, \boldsymbol{\nabla}_{\!R}\Theta).$$

Use the Clausius–Duhem inequality to show that the free energy is a function only of deformation and temperature,

$$\Psi = \Psi(\mathsf{F}, \Theta),$$

and that

$$\mathsf{T} = \rho\frac{\partial \Psi}{\partial \mathsf{F}}\mathsf{F}^T, \quad \eta = -\frac{\partial \Psi}{\partial \Theta}.$$

(ii) The specific heat capacity is defined to be

$$c(\mathsf{F}, \Theta) = -\Theta\frac{\partial^2 \Psi}{\partial \Theta^2} > 0.$$

Show that

$$c = \Theta\frac{\partial \eta}{\partial \Theta} = \frac{\partial \Phi}{\partial \Theta},$$

where $\Phi = \Psi + \eta\Theta$ is the internal energy.

(iii) For an entropic material c is assumed to be a function of Θ only. Use the results from (ii) to show that the change in internal energy for an entropic material caused by an increase in temperature from Θ_0 to Θ is given by

$$\Phi(\mathsf{F}, \Theta) - \Phi(\mathsf{F}, \Theta_0) = \int_{s=\Theta_0}^{s=\Theta} c(s)\mathrm{d}s,$$

and find a similar expression for the change in entropy, η.

(iv) Hence show that if $c(\Theta) = C$, a constant,

$$\Psi(\mathsf{F}, \Theta) = \Phi(\mathsf{F}, \Theta_0) - \Theta\eta(\mathsf{F}, \Theta_0) + F(\Theta),$$

where

$$F(\Theta) = C\left[\Theta - \Theta_0 - \Theta\ln\left(\frac{\Theta}{\Theta_0}\right)\right].$$

[18 marks]

FORMULA SHEET

- For a general (Lagrangian) coordinate system ξ^i:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \xi^i}, \quad \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}^j = \delta_i^j, \quad g_{ij} = \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}_j, \quad g = \det(g_{ij}).$$

$$\boldsymbol{G}_i = \frac{\partial \boldsymbol{R}}{\partial \xi^i}, \quad \boldsymbol{G}_i \boldsymbol{\cdot} \boldsymbol{G}^j = \delta_i^j, \quad G_{ij} = \boldsymbol{G}_i \boldsymbol{\cdot} \boldsymbol{G}_j, \quad G = \det(G_{ij}).$$

- For a scalar field $f(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$

$$\boldsymbol{\nabla} f = \boldsymbol{g}^i \frac{\partial f}{\partial \xi^i}, \quad \operatorname{div} \boldsymbol{u} = \frac{1}{\sqrt{g}} \frac{\partial \left(u^i \sqrt{g}\right)}{\partial \xi^i}, \quad \operatorname{curl} \boldsymbol{u} = \epsilon^{ijk} u_j|_i \boldsymbol{g}_k.$$

- The material derivative in general coordinates is

$$\frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + V^j U^i||_j,$$

where $\boldsymbol{V}$ is the velocity of the continuum.

- Cauchy's equation in the usual notation in components in general coordinates ξ^i is

$$T^{ji}||_j + \rho F^i = \rho \ddot{U}^i = \rho \frac{DV^i}{Dt}, \quad \text{where} \quad T^{ji}||_j = T^{ji}_{,j} + \Gamma^j_{jr} T^{ri} + \Gamma^i_{jr} T^{jr},$$

and Γ^i_{jk} are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

- The Reynolds Transport theorem states that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \phi \, \mathrm{d}\mathcal{V}_t = \int_{\Omega_t} \left(\frac{D\phi}{Dt} + \phi \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V} \right) \mathrm{d}\mathcal{V}_t,$$

where Ω_t is a material volume, ϕ is a scalar field and $\boldsymbol{V}$ is the velocity of the continuum.

- For a Cartesian line element $\mathrm{d}X_I$ in the deformed configuration

$$\frac{D\mathrm{d}X_I}{Dt} = V_{I,K} \mathrm{d}X_K,$$

where V_I is the I-th Cartesian component of the velocity.

- Nanson's relation states that

$$\mathrm{d}A_{\bar{i}} = J \frac{\partial \xi^j}{\partial \chi^{\bar{i}}} \mathrm{d}a_j,$$

where ξ^j are the Lagrangian coordinates, $\chi^{\bar{i}}$ are the Eulerian coordinates, J is the determinant of the deformation gradient tensor, $\mathrm{d}\boldsymbol{A}$ is an area element in the deformed configuration and $\mathrm{d}\boldsymbol{a}$ is an area element in the undeformed configuration.

- The Green–Lagrange strain tensor is defined by

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right).$$

- The strain invariants are defined by

$$I_1 = g^{ij}G_{ji}, \quad I_2 = \frac{1}{2}\left(I_1^2 - g^{ir}g^{js}G_{ij}G_{rs}\right), \quad I_3 = G/g,$$

where $g = det(g_{ij})$ and $G = det(G_{ij})$

- An incompressible hyperelastic material is described by a strain energy function $\mathcal{W}(I_1, I_2)$ such that

$$T^{ij} = PG^{ij} + Ag^{ij} + BB^{ij},$$

where

$$A = 2\frac{\partial \mathcal{W}}{\partial I_1}, \quad B = 2\frac{\partial \mathcal{W}}{\partial I_2} \quad \text{and} \quad B^{ij} = \left[I_1 g^{ij} - g^{ir}g^{js}G_{rs}\right].$$

- The Clausius–Duhem inequality is

$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta + \mathsf{T} : \mathsf{D} \geq 0.$$

END OF EXAMINATION PAPER