Three hours

THE UNIVERSITY OF MANCHESTER

CONTINUUM MECHANICS

21 January, 2014

14:00 – 17:00

Answer **ALL FOUR** questions in section A (21 marks in total).
Answer **THREE** of the **FOUR** questions in section B (54 marks in total). If more than THREE questions from Section B are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

A1.

(i) Show that the derivative of a vector $\boldsymbol{a}$ with respect to a general coordinate ξ^i is given by

$$\frac{\partial \boldsymbol{a}}{\partial \xi^i} = \frac{\partial (a^k \boldsymbol{g}_k)}{\partial \xi^i} = a^k|_i \, \boldsymbol{g}_k,$$

where

$$a^k|_i = \frac{\partial a^k}{\partial \xi^i} + \Gamma^k_{ij} a^j,$$

where $\boldsymbol{g}_k$ are the covariant base vectors associated with the coordinates ξ^k and Γ^k_{ij} are to be defined.

(ii) Show that $a^k|_i = 0$ does not imply that the components a^k are constant.

[4 marks]

A2. A continuum is deformed from an initial position $\boldsymbol{x}$ to $\boldsymbol{X}$ and the deformation gradient tensor is given by $\mathsf{F} = \partial \boldsymbol{X}/\partial \boldsymbol{x}$.

(i) If $\mathsf{U}^2 = \mathsf{F}^T\mathsf{F}$ and $\mathsf{R} = \mathsf{F}\mathsf{U}^{-1}$, show that R is orthogonal, $\mathsf{R}^T\mathsf{R} = \mathsf{I}$.

(ii) By computing the material derivative of the expressions $\mathsf{R}\mathsf{R}^T = \mathsf{I}$ and $\mathsf{F} = \mathsf{R}\mathsf{U}$, or otherwise, show that

$$\frac{D\mathsf{R}}{Dt} = \omega \mathsf{R},$$

where ω is a skew-symmetric tensor and

$$\frac{D\mathsf{U}}{Dt} = \mathsf{R}^T(\mathsf{L} - \omega)\mathsf{F},$$

where $\mathsf{L} = \boldsymbol{\nabla}_{\boldsymbol{R}} \otimes \boldsymbol{V}$ is the Eulerian velocity gradient tensor.

[7 marks]

A3. The centre of mass of a region Ω_t within the current configuration of a continuum is defined by

$$\boldsymbol{g}(t) = \frac{1}{m(\Omega_t)} \int_{\Omega_t} \rho(\boldsymbol{R}, t) \, \boldsymbol{R} \, \mathrm{d}\mathcal{V}_t,$$

where $m(\Omega_t)$ is the total mass in the region, ρ is the density and $\boldsymbol{R}$ is the position within the current configuration. Assuming that mass is conserved, show that

$$m(\Omega_t) \frac{D^2 \boldsymbol{g}}{Dt^2} = \frac{D\boldsymbol{P}}{Dt},$$

where $\boldsymbol{P}$ is the total momentum within the region. Interpret this result in the context of Newtonian particle mechanics.

[5 marks]

A4. A cube of material occupies the region $0 \le x_I \le 1$, $I = 1, 2, 3$, in a Cartesian coordinate system and has volume V_0. The cube undergoes a deformation such that

$$X_1 = (1 + \lambda_1)x_1, \quad X_2 = (1 + \lambda_2)x_2, \quad X_3 = (1 + \lambda_3)x_3.$$

The volume of the deformed block is given by V.

(i) Show that the fractional change in volume of the material is given by

$$\frac{V - V_0}{V_0} = I_1 + I_2 + I_3,$$

where I_1, I_2, I_3 are invariants of the infinitesimal (Cauchy) strain tensor defined by $e_{ij} = (u_i|_j + u_j|_i)/2$, The displacement vector field $\boldsymbol{u}$ is the difference between the deformed and original positions and $|$ represents the covariant derivative in the appropriate coordinate system.

(ii) What is the fractional volume change in terms of the invariants of the deformed metric tensor?

N.B. The three invariants of a second-rank tensor, A, are defined by

$$\det(\mathsf{A} - \mu \mathsf{I}) = -\mu^3 + I_1\,\mu^2 - \mu\, I_2 + I_3,$$

where I is the identity and μ is a scalar.

[5 marks]

SECTION B

B5. A hyperelastic material with strain energy function $\mathcal{W}$ moves from the undeformed position $\boldsymbol{x}$ to the deformed position $\boldsymbol{X}$. The components of the second Piola–Kirchhoff stress tensor in a general Lagrangian coordinate system ξ^i are given by

$$s^{ij} = a\, g^{ij} + b\, B^{ij} + p\, G^{ij}, \tag{1}$$

where

$$a = 2\frac{\partial \mathcal{W}}{\partial I_1}, \quad b = 2\frac{\partial \mathcal{W}}{\partial I_2}, \quad p = 2I_3\frac{\partial \mathcal{W}}{\partial I_3};$$
$$B^{ij} = I_1 g^{ij} - g^{ik} g^{jl} G_{kl};$$

I_1, I_2 and I_3 are invariants of the deformed metric tensor G_{ij}; and the undeformed metric tensor is g_{ij}.

(i) By transforming to Cartesian coordinates show that an alternative representation of the constitutive law is

$$\mathsf{s} = 2\left[\left(\frac{\partial \mathcal{W}}{\partial I_1} + I_1\frac{\partial \mathcal{W}}{\partial I_2}\right)\mathsf{I} - \frac{\partial \mathcal{W}}{\partial I_2}\,\mathsf{c} + I_3\frac{\partial \mathcal{W}}{\partial I_3}\,\mathsf{c}^{-1}\right],$$

where $\mathsf{c} = \mathsf{F}^T\mathsf{F}$ is the right Cauchy-Green deformation tensor and $\mathsf{F} = \partial\boldsymbol{X}/\partial\boldsymbol{x}$.

(ii) Find an equivalent expression for the Cauchy stress σ in terms of the left Cauchy-Green deformation tensor $\mathsf{b} = \mathsf{F}\mathsf{F}^T$.

You may assume that $\mathsf{s} = J\mathsf{F}^{-1}\sigma\mathsf{F}^{-T}$, without proof.

(iii) If the strain energy function is given by

$$\mathcal{W} = \frac{1}{2}(I_1 - 3 - 2\ln\sqrt{I_3}),$$

compute the Cauchy stress tensor corresponding to pure dilation, in which the volume increases uniformly by a factor of α. Find the value of α for which the stress is maximised.

[18 marks]

B6. An incompressible, generalised Oldroyd B fluid has the constitutive relationship

$$\mathsf{T} = -P\mathsf{I} + \widetilde{\mathsf{T}},$$

where P is the fluid pressure and the extra-stress, $\widetilde{\mathsf{T}}$, satisfies the relationship

$$\widetilde{\mathsf{T}} + \lambda_1 \widetilde{\mathsf{T}}^{\nabla} = \eta_0 \left(\mathsf{D} + \lambda_2 \mathsf{D}^{\nabla} \right),$$

where η_0, λ_1 and λ_2 are constants. The upper-convected derivative is defined by

$$\mathsf{A}^{\nabla} = \frac{D\mathsf{A}}{Dt} - \mathsf{L}\mathsf{A} - \mathsf{A}\mathsf{L}^T,$$

where $\mathsf{L} = \boldsymbol{\nabla}_{\boldsymbol{R}} \otimes \boldsymbol{V}$ is the Eulerian velocity gradient tensor and D is the symmetric part of L.

(i) Confirm that the constitutive relationships are objective.

You may assume that the Cauchy stress T is objective; that the deformation gradient tensor, F, transforms as $\mathsf{F}^* = \mathsf{Q}\mathsf{F}$ and $\dot{\mathsf{F}} = \mathsf{L}\mathsf{F}$, where Q is an orthogonal matrix that expresses the relative rotation between observers.

A steady two-dimensional extensional flow is given by $\boldsymbol{V} = (\epsilon X, -\epsilon Y)$, where ϵ is a constant rate of extension and X and Y are Cartesian coordinates.

(ii) Assuming that the extra stress is constant in time and space, use Cauchy's equation to find the Cauchy stress components T_{XX}, T_{YY}, T_{XY} of the fluid in this flow.

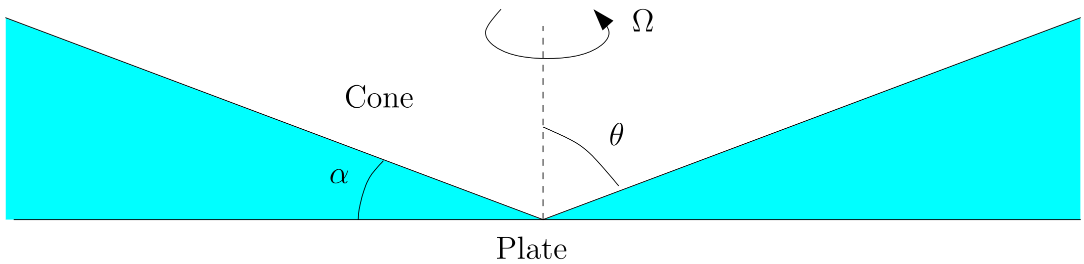
(iii) Comment on the validity of the constitutive model by considering what happens to the normal stress difference $T_{XX} - T_{YY}$ as $\epsilon \to 1/(2\lambda_1)$.

[18 marks]

B7. An incompressible, Newtonian fluid obeys the constitutive law

$$\mathsf{T} = -P\mathsf{I} + 2\mu\mathsf{D},$$

where P is the fluid pressure, T is the Cauchy stress, $\mathsf{D} = \frac{1}{2}\left(\boldsymbol{\nabla}\otimes\boldsymbol{V} + (\boldsymbol{\nabla}\otimes\boldsymbol{V})^T\right)$ is the rate of strain tensor, $\boldsymbol{V}$ is the fluid velocity and μ is a constant. The fluid is driven within a cone-and-plate device in which the cone rotates at a constant angular velocity Ω and the plate is fixed. The fluid occupies the region $\pi/2 - \alpha \leq \theta \leq \pi/2$, in a spherical polar coordinate system $(\xi^1, \xi^2, \xi^3) = (r, \theta, \phi)$, with origin at the point where the cone meets the plate.



(i) Write down the boundary conditions on the cone and plate. Hence, explain why $\boldsymbol{V} = r\sin\theta F(\theta)\boldsymbol{e}_\phi$ is a plausible form for the velocity field where $\boldsymbol{e}_\phi$ is a unit vector in the direction of rotation.

(ii) Let $\boldsymbol{V} = V^i\boldsymbol{g}_i$, where $\boldsymbol{g}_i$ are the covariant base vectors of the spherical polar coordinate system. Assuming that $\boldsymbol{V}$ has the form given in (i) find V^i in terms of $F(\theta)$.

(iii) By solving the component of Cauchy's equation corresponding to the ϕ direction, show that

$$T^{23} = \frac{f(r)}{\sin^3\theta},$$

and hence find an expression for $F(\theta)$ in terms of an integral that you need not evaluate.

You may use the fact that in spherical polar coordinates the position is given by

$$\boldsymbol{r} = r\sin\theta\cos\phi\boldsymbol{e}_x + r\sin\theta\sin\phi\boldsymbol{e}_y + r\cos\theta\boldsymbol{e}_z,$$

and the only non-zero Christoffel symbols are

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta},$$

$$\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r\sin^2\theta, \quad \Gamma^2_{33} = -\cos\theta\sin\theta.$$

[18 marks]

P.T.O.

B8. A solid material has a Helmholtz free energy per unit mass and heat flux per unit undeformed area of the forms

$$\psi(\mathsf{e}, \theta, \boldsymbol{\nabla}_{\!r}\theta, \gamma^1, \dots \gamma^m) \quad \text{and} \quad \boldsymbol{q}(\mathsf{e}, \theta, \boldsymbol{\nabla}_{\!r}\theta, \gamma^1, \dots \gamma^m),$$

where e is the Green–Lagrange strain tensor, θ is the temperature, $\boldsymbol{\nabla}_{\!r}\theta$ is the temperature gradient and γ^i are independent internal variables that are each symmetric, second-rank tensors.

(i) Use the Clausius–Duhem inequality in the form

$$-\rho_0\dot{\psi} - \rho_0\eta_0\dot{\theta} - \frac{1}{\theta}\boldsymbol{q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\!r}\theta + \mathsf{s} : \dot{\mathsf{e}} \geq 0.$$

to show that ψ cannot be a function of the temperature gradient and derive the relationships

$$\mathsf{s} = \rho_0\frac{\partial\psi}{\partial\mathsf{e}}, \quad \eta_0 = -\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad d = \sum_{\alpha=1}^{m}\chi^\alpha : \dot{\gamma}^\alpha \geq 0, \quad \text{where} \qquad \chi^\alpha = -\frac{\partial\psi}{\partial\gamma^\alpha},$$

for the second Piola–Kirchhoff stress tensor s, the entropy η_0 and dissipation, d.

We now assume that

$$\psi = \psi^\infty(\mathsf{e}, \theta) + \sum_{\alpha=1}^{m}\psi^\alpha(\mathsf{e}, \Theta, \gamma^\alpha),$$

where

$$\psi^\infty = \mu_\infty(\theta)\text{trace}[\mathsf{e}^2] \quad \text{and} \quad \psi^\alpha = \mu_\alpha(\theta)\text{trace}[(\mathsf{e} - \mathsf{\Lambda}^\alpha)^2], \quad \text{and} \quad \mathsf{\Lambda}^\alpha = \frac{1}{2}\left(\gamma^\alpha - \mathsf{I}\right),$$

and $\mu_\infty(\theta) > 0$ and $\mu_\alpha(\theta) > 0$.

(ii) Find the conditions on e, θ and γ^α that lead to zero free energy.

(iii) Compute s and η_0 and find the form of the dissipation for the given free energy function.

(iv) If we define $\mathsf{s}^\alpha = \rho_0\partial\psi^\alpha/\partial\mathsf{e}$ and insist that the dissipation is given by

$$d = \frac{1}{\rho_0}\sum_{\alpha=1}^{m}\frac{1}{\eta_\alpha}|\mathsf{s}^\alpha|^2,$$

find the relationship between s^α and $D\mathsf{\Lambda}^\alpha/Dt$.

[18 marks]

FORMULA SHEET

- Cauchy's equation in the usual notation in components in general coordinates ξ^i is

$$T^{ji}||_j + \rho F^i = \rho \ddot{U}^i = \rho \frac{DV^i}{Dt}, \quad \text{where} \quad T^{ji}||_j = T^{ji}_{,j} + \Gamma^j_{jr} T^{ri} + \Gamma^i_{jr} T^{jr},$$

  and Γ^i_{jk} are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

- The material derivative in general coordinates is

$$\frac{DV^i}{Dt} = \frac{\partial V^i}{\partial t} + V^j V^i|_j,$$

  where $|_j$ is the covariant derivative with respect to the given coordinates.

END OF EXAMINATION PAPER