MATH35021: Elasticity

Andrew Hazel
Andrew.Hazel@manchester.ac.uk

January 17, 2018

Chapter 0

Preliminaries

These notes cover the course MATH35021 (Elasticity) and are provided as a supplement to the lectures. The course does not follow any particular text, so you **should not need** to buy any text books. Hopefully, these notes are sufficiently self-contained that you will be able to use them to understand the course material. If you do wish to refer to textbooks the following is a list of books that you may find helpful.

Textbook which covers most of the material in this course:
Gould, P.L. *Introduction to Linear Elasticity, 2nd ed.* Springer (1994).

Nice (useful) review of Linear Algebra:
Banchoff, T. & Wermer, J. *Linear Algebra Through Geometry, 2nd ed.* Springer (1991).

One of the classic elasticity texts:
Green, A.E. & Zerna, W. *Theoretical Elasticity.* Dover (1992) – paperback reprint of the original version from Oxford University press.

And another classic:
Love, A.E.H. *Treatise on the Mathematical Theory of Elasticity.* Dover (1944) – paperback reprint of the original version from Cambridge University press.

A beautiful little book (but out of print!):
Long, R.R. *Mechanics of Solids and Fluids.* Prentice-Hall, (1961).

A complete treatment of the subject:
Wempner, G. *Mechanics of Solids with Applications to Thin Bodies.* Kluwer Academic Publishers Group (1982).

Another comprehensive doorstop:
Fung, Y. C. & Tong, P. *Classic and Computational Solid Mechanics* World Scientific Press (2001).

A more modern treatment, going beyond this course:
Howell, P., Kozyreff, G. & Ockendon, J *Applied Solid Mechanics* Cambridge University Press (2009).

0.1 Things you should already know

The course is as self-contained as it can be, but you should already be confident with the basic calculus of scalar and vector fields (div, grad, curl & multiple integrals, divergence theorem, ...); Taylor series for functions of many variables; the solution of ordinary and partial differential equations (general methods for linear equations); as well as basic linear algebra (how to work with matrices, vectors, definitions of eigenvalues, linear independence, ...). If you do not immediately know the answers to the questions in section 0.1.1 (or at least how to find the answers) then I would suggest revising the appropriate material. I will not assume any knowledge of mechanics beyond basic particle mechanics and Newton's laws, but, of course, if you have already done more advanced mechanics courses many of the concepts that we discuss should be familiar.

0.1.1 Pre-course fitness check

1. Two vectors are defined in components in a global Cartesian basis: $\boldsymbol{a} = (7, 8, 1)$, $\boldsymbol{b} = (3, 2, -1)$.
   (a) Find $\boldsymbol{a} \cdot \boldsymbol{b}$ and hence determine whether the two vectors are orthogonal.
   (b) Find two unit vectors $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ that are parallel to $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

2. Is it always possible to find the inverse of a 2×2 matrix? If so, prove it; if not, provide a counterexample. What about 3×3, or $n \times n$?

3. In Cartesian coordinates (x, y, z), $f = x + yz$ and $\boldsymbol{F} = (x^2, y, z)$.
   (a) Is f a scalar or vector field? What about $\boldsymbol{F}$?
   (b) Find $\boldsymbol{\nabla} f$, $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, and $\boldsymbol{\nabla} \times \boldsymbol{F}$.
   (c) What does the notation ∇^2 mean?

4. Find the Taylor series of $\cos(x + y)$ about the point $x = 0$, $y = \pi/2$.

5. A linear system of simultaneous equations is given by

$$\begin{aligned} 5x + 3y + 2z &= 2, \\ 2x + 7y &= 0, \\ 10x + 6y + 4z &= 7. \end{aligned}$$

   Write the system as a matrix equation. Does the system have a solution? If so, find the solution; if not, how could the system be changed to ensure that it does have a solution.

6. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 3 \\ 5 & 2 & 1 \end{pmatrix}.$$

7. Find the general solution $u(x)$ of the equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \omega^2 u = 0.$$

8. Find the solution $u(x, y)$ of the PDE $\nabla^2 u = 0$ in a square domain $x, y \in [0, 1]$, subject to the boundary conditions $u = 1$ on the line $y = 0$, but $u = 0$ on all other boundaries.

9. State Newton's three laws of motion. Use them to determine the position at which a cannonball fired at an angle of $\pi/4$ radians with a velocity of $1ms^{-1}$ returns to the ground, assuming uniform gravitational acceleration of magnitude g.

Chapter 1

Introduction

Lecture 1

The mathematical theory of elasticity concerns the development of models that describe the response of solid bodies to applied loads. In other words, we typically want to determine the deformation of a solid body and the internal forces within it, given the loads and displacements applied on the boundaries.

The theory has obvious applications to the strength of structures such as buildings, bridges, etc. as well as perhaps less obvious applications in seismology, biomechanics and plant growth among many others. There are three main ingredients required in any model in elasticity:

1. Characterize the deformation: **strain**.
2. Characterize the forces both external (**loads**) and internal (**stresses**).
3. Find the relationship between the deformation and forces: the **constitutive law**.

The strain and stress are characterized in the same way for any solid material; and the differences between materials arise only through the constitutive law. In this course we shall consider only the simplest possible constitutive law corresponding to an isotropic, linear elastic material.

Perfectly elastic behaviour means that all the deformations are reversible and that the stress depends only on the current state of strain. In other words there are no history or memory effects and no permanent deformation. Materials that retain some memory of their previous stress states are known as viscoelastic and the theory of permanent deformation is called plasticity. All materials will exhibit elastic behaviour under small deformations, but above a critical strain the material will yield plastically and not return to its original shape. For example, consider a metal paper clip: if you pull the end a little it will spring back into shape, but if you bend the end out, then it will remain in the new shape.

1.1 Index notation & summation convention

In order to develop a general theory of elasticity we will need to work with vectors because we live in (at least) three spatial dimensions. We can keep the notation compact by using index notation, in which we represent a vector by its components in a Cartesian (orthonormal) coordinate system. The vector $\boldsymbol{r} = (r_1, r_2, r_3)$ is represented by writing r_i where i, the index, takes the value 1, 2 or 3. We will also need to work with tensor quantities which are represented by matrices, with two indices:

$$\mathsf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

can be represented by $A_{\alpha\beta}$ where α and β take the values 1 or 2. We shall adopt the convention that Greek indices (α, β, γ, ...) always range from 1 to 2; and Latin indices (i, j, k, ...) range from 1 to 3.

One advantage of index notation is that it is completely explicit. Consider the dot product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we can write this is $\boldsymbol{a} \cdot \boldsymbol{b}$ and we know (or should know) that the result is a scalar, but how do we compute it? Using index notation we can write

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i \equiv a_i b_i,$$

where the last term uses the *summation convention* that a repeated index is summed over all values of the index; *i.e.* we "drop the Σ's". Similarly, a matrix vector product $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ in index notation is

$$\sum_{\beta=1}^{2} A_{\alpha\beta} x_\beta = b_\alpha \quad \text{or using summation convention} \quad A_{\alpha\beta} x_\beta = b_\alpha.$$

The above expression represents two different equations because there is a so-called free (not summed) index α.

Notes

- The repeated index is a dummy variable because it simply represents a summation, so it can be changed when convenient

$$\sum_{i=1}^{3} a_i b_i \equiv a_i b_i = a_j b_j = a_k b_k \equiv \sum_{k=1}^{3} a_k b_k.$$

- An index can **never** appear more than twice in a single term

$$a_i b_i c_i \quad \text{is ILLEGAL.}$$

The problem here is that if an index is repeated three times then the meaning is potentially ambiguous. It could mean

$$\sum_{i=1}^{3} a_i b_i c_i \quad \text{or} \quad \left(\sum_{i=1}^{3} (a_i b_i) \right) c_i,$$

or something else. Thus, we do not allow such a construction.

- Any free (non-repeated) indices must balance — appear in every term in an equation:

$$a_i = b_i + c_i, \qquad A_{ij} = C_{ik} D_{kj}.$$

The following construction is illegal

$$a_i = A_{ij},$$

because it states that a vector is equal to a rank-two matrix, which cannot be the case. The equation $A_{ij} = B_{jk}$ is also illegal because the free indices are not the same on both sides of the equation.

- In certain circumstances we may wish to repeat an index but not sum over it. In those cases, we will use brackets $e_{(i)(i)}$ means the terms e_{11}, e_{22} or e_{33} and NOT their sum.

The Kronecker Delta is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

It is "essentially" the identity matrix, but it can be used to exchange indices

$$a_i b_j \delta_{ij} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \delta_{ij} = a_1 b_1 \delta_{11} + a_1 b_2 \delta_{12} + a_1 b_3 \delta_{13} + a_2 b_1 \delta_{21} + a_2 b_2 \delta_{22} + a_2 b_3 \delta_{23} + a_3 b_1 \delta_{31} + a_3 b_2 \delta_{32} + a_3 b_3 \delta_{33},$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i = a_j b_j.$$

In other words, if there is a sum over one of the indices of the Kronecker Delta the other occurrence of the index can be replaced by the remaining index in the Kronecker Delta: $a_i \delta_{ij} = a_j$. Note that $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$.

To further save on writing, a comma is used to denote partial differentiation

$$\frac{\partial a}{\partial x_i}, \qquad \frac{\partial u_i}{\partial x_j} = u_{i,j}.$$

This means that we can express common differential operators in index notation:

$$\nabla \cdot \mathbf{u} = \text{div } \mathbf{u} = u_{i,i} \tag{1.1}$$

$$\nabla \phi = \text{grad}\phi = \phi_{,i} \tag{1.2}$$

$$\nabla^2 \phi = \phi_{,ii} \tag{1.3}$$

Finally, we note that because Cartesian components are independent

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

Chapter 2

Analysis of strain

Strain quantifies the deformation of a solid body. It is important to realise that a solid body can move without deformation: simple translations and rotations are known as rigid-body motions and do not induce any deformation. Therefore any measure of strain should be zero for such motions. Throughout this course we shall consider only infinitesimal deformations. Lecture 2

2.1 The infinitesimal (Cauchy) strain tensor

In order to actually quantify deformation we require a mathematical framework that allows us to track the motion of individual points within the body. The idea is simply that we mark each individual point in the body and follow its motion. Consider the situation shown in Figure 2.1.

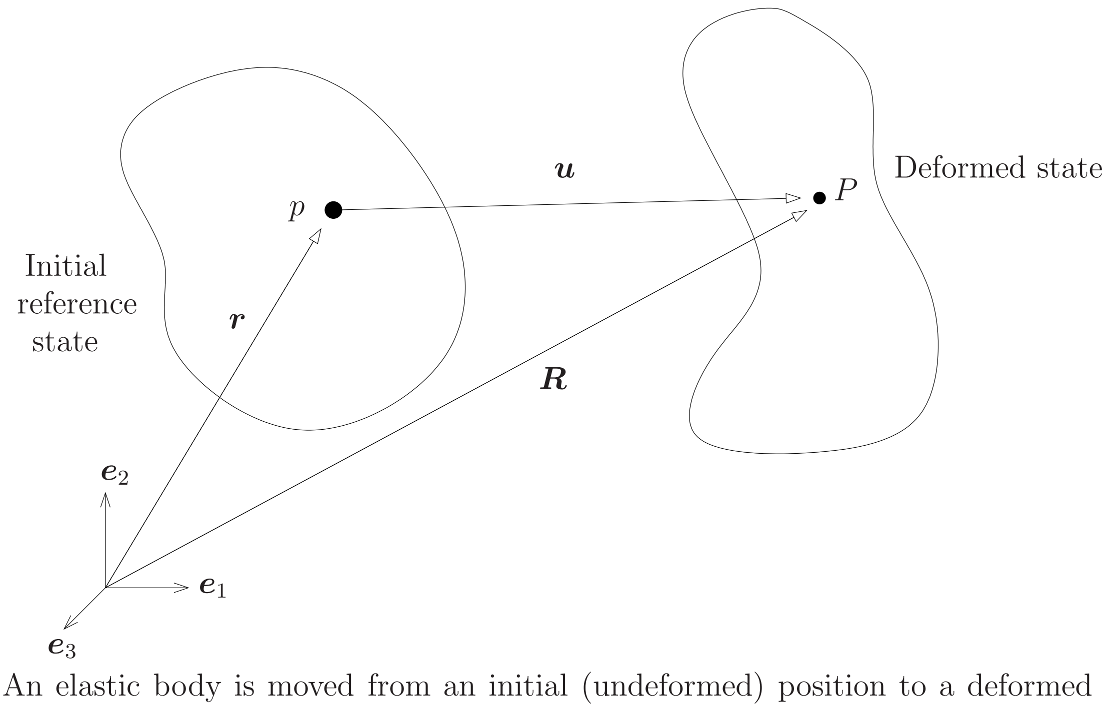


Figure 2.1: An elastic body is moved from an initial (undeformed) position to a deformed state. Imagine that we mark a point on the on the undeformed body, p, and describe it by a position vector $\boldsymbol{r}$ from a chosen origin. In the deformed position the *same material* point is now denoted P and is described by the position vector $\boldsymbol{R}$. The displacement of the point $\boldsymbol{u}$ is given by $\boldsymbol{R} - \boldsymbol{r}$.

If we have a single marked point whose initial position vector is given by $\boldsymbol{r}$ then its deformed position $\boldsymbol{R}$ can be decomposed into the initial position and a displacement vector:

$$\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{u}. \tag{2.1}$$

Note that in general $\boldsymbol{u}$ will vary with position and so we use a Lagrangian description in which $\boldsymbol{u}(\boldsymbol{r})$ is treated as a function of the undeformed position[1].

Note also that we shall use $\boldsymbol{r}$ and $\boldsymbol{x}$ interchangeably to represent the position vector. (I know that's annoying, but all the textbooks do this as well!) Thus, we can write the following equivalent statements in vector and index notation

$$\begin{aligned}
\boldsymbol{R} &= \boldsymbol{r} + \boldsymbol{u} \\
\boldsymbol{R} &= \boldsymbol{x} + \boldsymbol{u} \\
\boldsymbol{X} &= \boldsymbol{x} + \boldsymbol{u} \\
\boldsymbol{R}(\boldsymbol{x}) &= \boldsymbol{r} + \boldsymbol{u}(\boldsymbol{x}) \\
R_i(x_j) &= r_i + u_i(x_j) \\
R_i(x_j) &= x_i + u_i(x_j)
\end{aligned}$$

Example 2.1. Deformation of a unit square
A unit square $x_1 \in [0,1]$, $x_2 \in [0,1]$ has the undeformed position vector $\boldsymbol{r} = (r_1, r_2) = (x_1, x_2)$. If the displacement vector is $\boldsymbol{u} = (x_1, 3 + x_2)$ sketch the undeformed and deformed positions.

Solution 2.1.

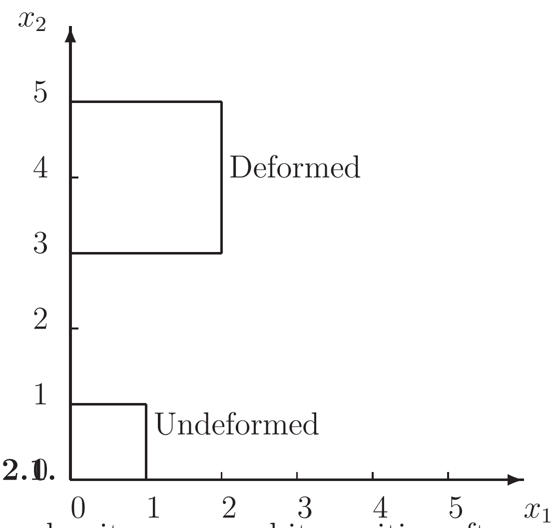


Figure 2.2: The undeformed unit square and its position after application of the displacement field $\boldsymbol{u} = (x_1, 3 + x_2)$.

We know that the deformed position is given by

$$\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{u},$$

$$\begin{pmatrix} R_1(x_1, x_2) \\ R_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ 3 + x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 + 3 \end{pmatrix}.$$

---

[1]The alternative Eulerian description in which $\boldsymbol{u}(\boldsymbol{R})$ is treated as a function of the deformed position is actually identical in our linear theory because in a linear approximation using Taylor series we can write $R_i \approx r_i + R_{i,j} r_j$, so that

$$u_i(R_k) = u_i(r_k) + u_{i,m}(r_k) R_{m,l} r_l \approx u_i(r_k).$$

In the above the gradient terms are taken with respect to the undeformed coordinates and we assume they are $\mathcal{O}(\epsilon)$ so that their product is negligible.

Hence, we have the deformed position as a function of the undeformed coordinates (x_1, x_2) which both take the values in the range 0 to 1. Consider the deformation of the corners. The point originally at $(x_1 = 0, x_2 = 0)$ maps to the point $\boldsymbol{R}(0,0) = (2 \times 0, 2 \times 0 + 3) = (0,3)$. Similarly, the point originally at $(1,0)$ maps to $(2,3)$; the point originally at $(0,1)$ maps to $(0,5)$ and the point originally at $(1,1)$ maps to $(2,5)$. In this case the displacement vector is linear so we simply connect the corners by straight lines to find the deformed configuration.

We now have enough machinery to quantify the motion of our material, but how can we tell whether it has deformed or not. It turns out that simply looking at the motion of individual points does not give us enough information. Instead, we consider the distances between material points or line elements. A body will be strained if the distance between any two material points has changed. Let us mark a second point, q, on our imaginary elastic body a small distance from the original point p, see Figure 2.3

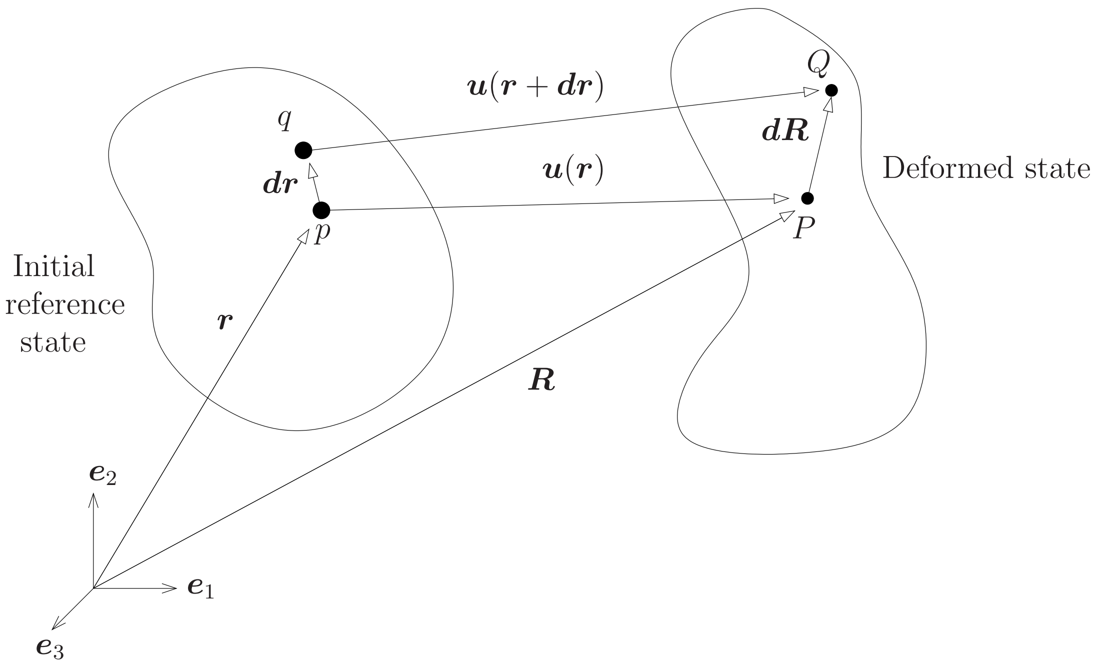


Figure 2.3: Two points p and q are marked on an elastic body that is subsequently deformed. In order to determine whether or not the body is strained we must compare the lengths of the line elements between the undeformed and deformed positions of the points: $\boldsymbol{dr}$ and $\boldsymbol{dR}$, respectively.

We can determine whether or not the material is strained by comparing the vector line elements $\boldsymbol{dr}$ and $\boldsymbol{dR}$ between the two marked points in the undeformed and deformed configurations, respectively. We need to establish the connection between the two line elements, which follows from vector geometry, see Figure 2.3:

$$\boldsymbol{dR} = \boldsymbol{u}(\boldsymbol{r} + \boldsymbol{dr}) - \boldsymbol{u}(\boldsymbol{r}) + \boldsymbol{dr}. \tag{2.2}$$

Writing equation (2.2) in index notation gives

$$dR_i = dr_i + u_i(r_j + dr_j) - u_i(r_j). \tag{2.3}$$

Note that we must use a different index in the argument of $\boldsymbol{u}$ to indicate that all components of the displacement vector u_i are potentially functions of all components of the position vector, r_j. We must also use a different index to avoid accidental summation when none is implied.

We now use Taylor series to expand the second term in equation (2.3)

$$dR_i = dr_i + u_i(r_j) + \left.\frac{\partial u_i}{\partial r_j}\right|_{\boldsymbol{r}} dr_j + \cdots - u_i(r_j), \tag{2.4}$$

and neglect the non-linear terms under the assumption that the line element $\boldsymbol{dr}$ is small, $|\boldsymbol{dr}| \ll 1$. Note that the summation convention has been used in the term involving partial derivatives which must be summed over all position vector components[2]. Hence, we can write

$$dR_i \approx dr_i + \left.\frac{\partial u_i}{\partial r_j}\right|_{\boldsymbol{r}} dr_j = dr_i + \left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{r}} dr_j. \tag{2.5}$$

The equation (2.5) relates an infinitesimal line element in the deformed configuration to its original position in the reference configuration. The first term corresponds to a rigid body translation which does not change the length or direction of the line element. The second term contains all information about the deformation, but also rigid body rotations. The quantity $\frac{\partial u_i}{\partial x_j}$ is called the *displacement gradient tensor*. It can be written in a coordinate independent way as $\boldsymbol{\nabla} \otimes \boldsymbol{u}$, where $\otimes$ indicates the tensor product. (The tensor product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{a} \otimes \boldsymbol{b}$: a rank-two tensor with components $a_i b_j$.)

Although useful, equation (2.5) does not provide enough information to determine whether or not deformation has taken place. Instead, we must compare the two lengths $|\boldsymbol{dR}|$ and $|\boldsymbol{dr}|$. It's actually slightly easier to compare $|\boldsymbol{dR}|^2$ and $|\boldsymbol{dr}|^2$, which does not change our conclusions because x^2 is a monotonic function. The reason why it is easier to compare the square lengths is because we can simply use the dot product of the vector with itself: $|\boldsymbol{dR}|^2 = dR_i dR_i$. Now using equation (2.5), we obtain

$$|\boldsymbol{dR}|^2 = \left(dr_i + \left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{r}} dr_j\right)\left(dr_i + \left.\frac{\partial u_i}{\partial x_k}\right|_{\boldsymbol{r}} dr_k\right).$$

Note that we have used a different dummy index in the second expression in parentheses to avoid violating the summation convention when multiplying out the brackets. We can take out common factors by using the Kronecker delta

$$|\boldsymbol{dR}|^2 = \left(\delta_{ij} + \left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{r}}\right) dr_j \left(\delta_{ik} + \left.\frac{\partial u_i}{\partial x_k}\right|_{\boldsymbol{r}}\right) dr_k,$$

expand the brackets

$$|\boldsymbol{dR}|^2 = \left(\delta_{ij}\delta_{ik} + \delta_{ik}\left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{r}} + \delta_{ij}\frac{\partial u_i}{\partial x_k}\right) dr_j dr_k,$$

and neglect the product of displacement gradient tensors because we are working with linear elasticity and therefore neglect all quadratic terms. (In other words, we assume that the displacement

[2] If you are unsure about this consider the Taylor series expansion of a scalar function of two variables $f(x_1, x_2)$ about $(0,0)$:

$$f(0,0) + \left.\frac{\partial f}{\partial x_1}\right|_{(0,0)} x_1 + \left.\frac{\partial f}{\partial x_2}\right|_{(0,0)} x_2 + \cdots = f(0,0) + \frac{\partial f}{\partial x_j} x_j + \cdots$$

gradient is so small that any products are negligible, $\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$.) Thus, after using the properties of the Kronecker delta we obtain

$$|\boldsymbol{dR}|^2 = \left(\delta_{jk} + \left.\frac{\partial u_k}{\partial x_j}\right|_{\boldsymbol{r}} + \left.\frac{\partial u_j}{\partial x_k}\right|_{\boldsymbol{r}} \right) dr_j dr_k. \tag{2.6}$$

We now include the length of the original line element by using the classic trick of writing a vector as its length multiplied by a unit vector in the appropriate direction $dr_i = |\boldsymbol{dr}| n_i$, where $\boldsymbol{n}$ is a unit vector. Equation (2.6) becomes

$$|\boldsymbol{dR}|^2 = \left[\delta_{jk} n_j n_k + \left(\left.\frac{\partial u_k}{\partial x_j}\right|_{\boldsymbol{r}} + \left.\frac{\partial u_j}{\partial x_k}\right|_{\boldsymbol{r}} \right) n_j n_k \right] |\boldsymbol{dr}|^2,$$

$$= \left[n_j n_j + \left(\left.\frac{\partial u_k}{\partial x_j}\right|_{\boldsymbol{r}} + \left.\frac{\partial u_j}{\partial x_k}\right|_{\boldsymbol{r}} \right) n_j n_k \right] |\boldsymbol{dr}|^2,$$

$$\Rightarrow \quad |\boldsymbol{dR}|^2 = |\boldsymbol{dr}|^2 \left[1 + \left(\left.\frac{\partial u_k}{\partial x_j}\right|_{\boldsymbol{r}} + \left.\frac{\partial u_j}{\partial x_k}\right|_{\boldsymbol{r}} \right) n_j n_k \right]. \tag{2.7}$$

The first term in square brackets is one because $\boldsymbol{n}$ is a unit vector so that $n_j n_j = \boldsymbol{n} \cdot \boldsymbol{n} = 1$. Hence, the deformation, if any, is contained entirely in the term in parenthesis in equation (2.7) — the symmetric part of the displacement gradient tensor.

Lecture 3

It is always possible to write a tensor (matrix) as the sum of symmetric and antisymmetric parts, *e.g.*

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + w_{ij},$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \qquad w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right).$$

It is straightforward to show that $e_{ij} = e_{ji}$ (symmetric) and that $w_{ij} = -w_{ji}$ (antisymmetric).

From equation (2.7) we can write

$$|\boldsymbol{dR}|^2 = |\boldsymbol{dr}|^2 \left[1 + 2 e_{jk} n_j n_k \right], \tag{2.8}$$

and, therefore, e_{ij} is called the (infinitesimal or Cauchy) strain tensor. For reasons that will soon become clear w_{ij} is called the rotation tensor. We can write equation (2.5) in the form

$$\begin{array}{cccccc} dR_i & = & dr_i \quad + & w_{ij}\, dr_j \quad + & e_{ij}\, dr_j, & \\ & & \text{Rigid-Body} & \text{Rigid-body} & \text{Pure Deformation} & (2.9) \\ & & \text{Displacement} & \text{Rotation} & \text{(Strain)} & \end{array}$$

which illustrates the decomposition into rigid-body motion and actual deformation.

2.2 Rigid-body rotation ($e_{ij} = 0$)

The rotation tensor is antisymmetric $w_{ij} = -w_{ji}$ which means that the diagonal components must be zero, because $w_{11} = -w_{11} = 0$ and so on. Hence, there are only three independent components of w_{ij} and

$$w_{ij}dr_j = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} dr_1 \\ dr_2 \\ dr_3 \end{pmatrix}, \tag{2.10}$$

where the labels for the components of the tensor have been carefully chosen so that we can write

$$w_{ij}dr_j = [\boldsymbol{\omega} \times \boldsymbol{dr}]_i , \quad \text{where} \quad \boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}. \tag{2.11}$$

In the above $\times$ is the standard vector (cross) product.

If we assume that there is no strain e_{ij} then using equations (2.11) and (2.9) we have

$$\boldsymbol{dR} = \boldsymbol{dr} + \boldsymbol{\omega} \times \boldsymbol{dr},$$

which can be represented graphically, see Figure 2.4. If $|\boldsymbol{\omega}| \ll 1$ then $|\boldsymbol{dr}| \approx |\boldsymbol{dR}|$ and so the motion

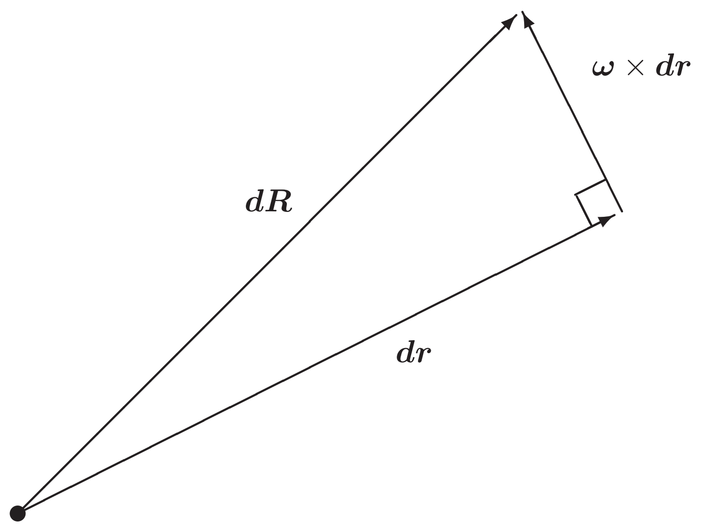


Figure 2.4: The rotation vector $\boldsymbol{\omega}$ is directed out of the page towards the reader. The cross product $\boldsymbol{\omega} \times \boldsymbol{dr}$ is orthogonal to both $\boldsymbol{dr}$ and $\boldsymbol{\omega}$ and hence, for small enough magnitudes of $\boldsymbol{\omega}$ the motion is a rigid-body rotation.

is a rigid-body rotation.

Thus the entries of rotation tensor w_{ij} describe the vector about which any rigid-body rotation is taking place.

2.3 Pure deformation

2.3.1 Extensional deformation

We have already established, equation (2.8), that

$$|\boldsymbol{dR}|^2 = |\boldsymbol{dr}|^2 \left[1 + 2e_{ij}n_in_j\right],$$

and we note that the term $2e_{ij}n_in_j$ is simply a scalar. We now adopt the common notation that $|\boldsymbol{dR}| = dS$ and $|\boldsymbol{dr}| = ds$, and write

$$dS = ds\sqrt{1 + 2e_{ij}n_in_j},$$

where the positive branch is taken because lengths must be positive. Remember that we are always assuming that displacement gradients are small, which means that both strains and rotations are small. Hence we can use the series expansion

$$\sqrt{1+2x} = (1+2x)^{1/2} = 1 + x + \ldots,$$

to write

$$dS \approx ds\left(1 + e_{ij}n_in_j\right), \qquad |e_{ij}| \ll 1.$$

Thus, the relative extension of the line element $\boldsymbol{dr} = ds\boldsymbol{n}$ is given by

$$\frac{\text{New length} - \text{Old length}}{\text{Old length}} = \frac{dS - ds}{ds} = e_{ij}n_in_j, \tag{2.12}$$

which determines the normal strain, e_n, also called the extension ratio, in the direction given by the unit vector $\boldsymbol{n}$. If the unit vector $\boldsymbol{n}$ is chosen to be a unit vector in the i-th Cartesian coordinate direction then $\boldsymbol{n} = \boldsymbol{e}_i$. For $\boldsymbol{n} = \boldsymbol{e}_1 = (1, 0, 0)$, the normal strain is $e_n = e_{11}$ and so e_{11} is the relative extension in the x_1 direction. Thus, $e_{(i)(i)}$ (no summation) is the normal strain in the i-th coordinate direction, which provides an interpretation for the diagonal terms in the strain tensor.

2.3.2 Shear deformation

We now know that the diagonal terms in the strain tensor describe the relative extensions in the Cartesian coordinate directions, but what do the off-diagonal terms mean?

Let us consider two different line elements

$$dr_i^{(1)} = ds^{(1)}n_i^{(1)}, \qquad dr_i^{(2)} = ds^{(2)}n_i^{(2)},$$

where the raised number in brackets is simply used to distinguish the two. In the undeformed configuration the two line elements are orthogonal, so that $\boldsymbol{dr}^{(1)}\boldsymbol{\cdot}\boldsymbol{dr}^{(2)} = 0$. Under the deformation the two line elements will change position and orientation and may not remain orthogonal, see Figure 2.5.

What is the angle, ϕ, between the deformed line elements?

We can answer this by considering the dot product $\boldsymbol{dR}^{(1)}\boldsymbol{\cdot}\boldsymbol{dR}^{(2)}$, which by equation (2.5) becomes

$$\boldsymbol{dR}^{(1)}\boldsymbol{\cdot}\boldsymbol{dR}^{(2)} = \left(dr_i^{(1)} + \left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{r}} dr_j^{(1)}\right)\left(dr_i^{(2)} + \left.\frac{\partial u_i}{\partial x_k}\right|_{\boldsymbol{r}} dr_k^{(2)}\right),$$

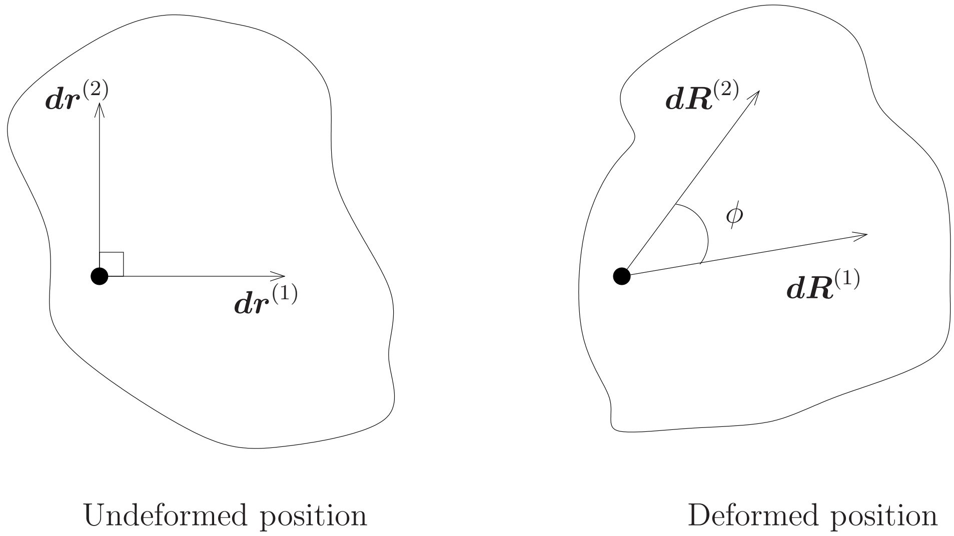


Figure 2.5: Two line elements that were initially orthogonal are carried to new positions under deformation.

$$= dr_i^{(1)}dr_i^{(2)} + \left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{r}} dr_j^{(1)}dr_i^{(2)} + \left.\frac{\partial u_i}{\partial x_k}\right|_{\boldsymbol{r}} dr_k^{(2)}dr_i^{(1)} + \mathcal{O}\left[\left(\frac{\partial u}{\partial x}\right)^2\right],$$

$$\approx \left(\left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{r}} + \left.\frac{\partial u_j}{\partial x_i}\right|_{\boldsymbol{r}}\right) dr_i^{(1)}dr_j^{(2)},$$

because the first term is $\boldsymbol{dr}^{(1)}\boldsymbol{\cdot}\boldsymbol{dr}^{(2)} = 0$ by construction.

From the definition of the dot product

$$\boldsymbol{dR}^{(1)}\boldsymbol{\cdot}\boldsymbol{dR}^{(2)} = dS^{(1)}dS^{(2)}\cos\phi = \left(\left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{r}} + \left.\frac{\partial u_j}{\partial x_i}\right|_{\boldsymbol{r}}\right) dr_i^{(1)}dr_j^{(2)},$$

from above. Thus,

$$dS^{(1)}dS^{(2)}\cos\phi = 2e_{ij}n_i^{(1)}n_j^{(2)}ds^{(1)}ds^{(2)},$$

and keeping only leading order terms we have $dS^{(1)} \approx ds^{(2)}$ and $dS^{(2)} \approx ds^{(2)}$, which then cancels from both sides to give

$$\cos\phi \approx 2e_{ij}n_i^{(1)}n_j^{(2)}. \tag{2.13}$$

In particular, if we pick $\boldsymbol{n}^{(1)} = \boldsymbol{e}_1 = (1,0,0)$ and $\boldsymbol{n}^{(2)} = \boldsymbol{e}_2 = (0,1,0)$, then $\cos\phi = 2e_{12}$.

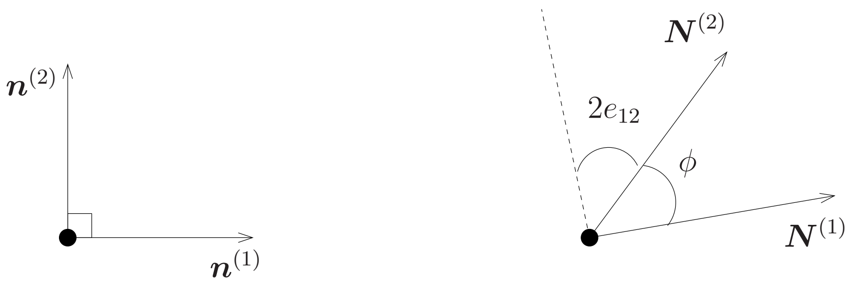


Figure 2.6: The off-diagonal term e_{12} in the strain tensor is half the angular contraction between two unit vectors originally in the x_1 and x_2 directions.

By construction we are assuming the the deformations are small, so the change in angle between the two line elements from $\pi/2$ must remain small. Hence,

$$2e_{12} = \cos\phi = \sin(\pi/2 - \phi) \approx \pi/2 - \phi,$$

and we conclude that e_{12} is half the angular contraction between two line elements originally in the x_1 and x_2 directions, see Figure 2.6. The off-diagonal components of the strain tensor, e_{ij} $i \neq j$, are called the shear strains. We now have a complete interpretation of all terms appearing in the strain and rotation tensors.

2.4 Principal strains and principal axes of strain

The normal strain at a point is defined by equation (2.12) Lecture 4

$$e_n = e_{ij} n_i n_j$$

and is a function of the unit vector $\boldsymbol{n}$. We can ask the simple(?) question: what is the maximum normal strain at a given location? This is actually a constraint optimisation problem because the vector $\boldsymbol{n}$ must be of unit length.

$$\max \; e_n(n_1, n_2, n_3) \quad \text{with constraint} \quad |\boldsymbol{n}| = 1.$$

The appropriate method to solve such problems is to use Lagrange multipliers

$$\max \; L = \left[e_{ij} n_i n_j + \lambda \left(1 - n_k n_k \right) \right],$$

where λ is the Lagrange multiplier associated with the normalisation constraint.

At an extremum,

$$\frac{\partial L}{\partial n_k} = 0 = e_{ij} \delta_{ik} n_j + e_{ij} n_i \delta_{jk} - 2\lambda n_k,$$
$$\Rightarrow e_{kj} n_j + e_{ik} n_i - 2\lambda n_k = 0.$$

Using the symmetry of the strain tensor and relabelling the repeated (dummy) index gives

$$2 e_{kj} n_j - 2\lambda n_k = 0 \quad \Rightarrow \quad \left(e_{kj} - \lambda \delta_{kj} \right) n_j = 0. \tag{2.14}$$

In matrix-vector form, the above equation is $(E - \lambda I)\boldsymbol{n} = \boldsymbol{0}$. In other words, the direction $\boldsymbol{n}$ in which the normal strain is maximised is an *eigenvector* of the strain tensor e_{ij}. Taking the dot product of equation (2.14) with $\boldsymbol{n}$ yields

$$e_{kj} n_j n_k - \lambda \delta_{kj} n_j n_k = 0 \quad \Rightarrow \quad e_n - \lambda = 0 \quad \Rightarrow \quad \lambda = e_n.$$

Thus, the associated eigenvalue is the corresponding normal strain.

In order to find the maximum normal strain, we simple compute the eigenvalues of the strain tensor and choose the greatest. Conversely, the minimum normal strain is the smallest eigenvalue of strain tensor. The following definitions are motivated by the above discussion.

- The principal axes of strain are the (normalised) eigenvectors of e_{ij}.
- The eigenvalues of e_{ij} are the *principal strains*, i.e. the strains in the directions of the normal axes.

Note that the strain tensor e_{ij} is symmetric and so the eigenvectors are orthogonal and provide a basis in which the strain tensor is diagonal. Note also that the principal axes and strains will, in general, vary from point to point throughout the body.

2.4.1 Strain Invariants

There are three strain invariants (quantities that do not change if the axes that define the strain tensor are rotated). The invariants are the coefficients of the cubic characteristic equation

$$\det(e_{ij} - \lambda\delta_{ij}) = 0.$$

If the coefficients were not invariant then the eigenvalues would change under rotation, which would mean that the strains within the body would change if we change coordinates (this can be interpreted as simply looking at the material from a different direction), which can't possibly be true!

Expanding the determinant by rows and using symmetry gives

$$-\lambda^3 + (e_{11} + e_{22} + e_{33})\lambda^2 + (e_{12}e_{12} + e_{13}e_{13} + e_{23}e_{23} - e_{11}e_{22} - e_{22}e_{33} - e_{33}e_{11})\lambda + \det e_{ij} = 0.$$

Thus the three invariants are:

– **the dilation:** $d = e_{ii}$ which represents the local fractional volume expansion (in the linear approximation)

$$d = e_{ii} = (dV - dv)/dv, \tag{2.15}$$

where dv is the volume of an infinitesimal region in the undeformed configuration and dV is the corresponding volume after deformation;

– **the determinant:** $\det e_{ij}$;

– **and a third quantity:** $1/2(e_{ij}e_{ij} - e_{ii}e_{jj})$.

Summary

For a small deformation the deformation field in the vicinity of a point $\boldsymbol{r}$ always consists of

- a rigid-body translation,
- a rigid-body rotation,
- three mutually orthogonal stretches (extensions).

Note that any of these components may be zero and, in general, the stretches will change in direction and magnitude as we move through the body.

Example

In order make this all a bit more concrete we shall consider an extended example that will use the developed theory. A two-dimensional elastic body is subject to the displacement field $\boldsymbol{u} = \epsilon(x_2, x_1)$, *i.e.* $u_1 = \epsilon x_2$ and $u_2 = \epsilon x_1$ where $\epsilon \ll 1$. Hence, the deformed position is given by Lecture 5

$$\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \epsilon x_2 \\ \epsilon x_1 \end{pmatrix} = \begin{pmatrix} x_1 + \epsilon x_2 \\ x_2 + \epsilon x_1 \end{pmatrix}.$$

If the elastic body is a unit square $x_1, x_2 \in [0, 1]$ in the undeformed configuration we wish to answer three questions:

1. What is the strain along the diagonal?

2. What is the change in angle between the lines originally along $(0, 1)$ and $(1, 0)$ emanating from the origin?

3. What are the principal strains and principal axes of strain?

For the first question let's use the general theory:

- Compute the displacement gradient tensor:

$$\frac{\partial u_1}{\partial x_1} = 0, \quad \frac{\partial u_1}{\partial x_2} = \epsilon, \quad \frac{\partial u_2}{\partial x_1} = \epsilon, \quad \frac{\partial u_2}{\partial x_2} = 0.$$

  Thus, setting these terms out as a matrix we can write

$$\frac{\partial u_i}{\partial x_j} = \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- Compute the strain and rotation tensors:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{(symmetric part of displacement-gradient tensor).}$$

  We can apply the formula, or recognise that, in this example, the displacement gradient tensor is already symmetric so that

$$\Rightarrow e_{ij} = \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  The rotation tensor is the antisymmetric part of the displacement gradient tensor. Again would could apply the formula, or recognise that it must be zero:

$$w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- Compute the strain in a particular direction. We need a unit normal in the chosen direction in the original (undeformed) coordinates. For the leading diagonal the direction is given by $(1, 1)$ and so our required unit vector is $\boldsymbol{n} = (\sqrt{2}/2, \sqrt{2}, 2)$. We then use the formula (2.12)

$$e_n = e_{ij}n_i n_j = \left(\sqrt{2}/2, \sqrt{2}/2\right) \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} = \epsilon.$$

  Thus the strain along the leading diagonal is ϵ.

We could also calculate this strain from first principles. Consider the original and deformed configurations, see Figure 2.7. The length of both diagonals of the original unit square is $\Delta s = \sqrt{2}$. In the deformed configuration the diagonal originally along the direction $(1, 1)$ remains in that direction and its length follow directly from the Pythagoras theorem

$$\Delta S = \sqrt{(1+\epsilon)^2 + (1+\epsilon)^2} = \sqrt{2}\sqrt{(1+\epsilon)^2} = \sqrt{2}(1+\epsilon).$$

Thus the strain (relative extension) is

$$\frac{\Delta S - \Delta s}{\Delta s} = \frac{\sqrt{2}(1+\epsilon) - \sqrt{2}}{\sqrt{2}} = 1 + \epsilon - 1 = \epsilon,$$

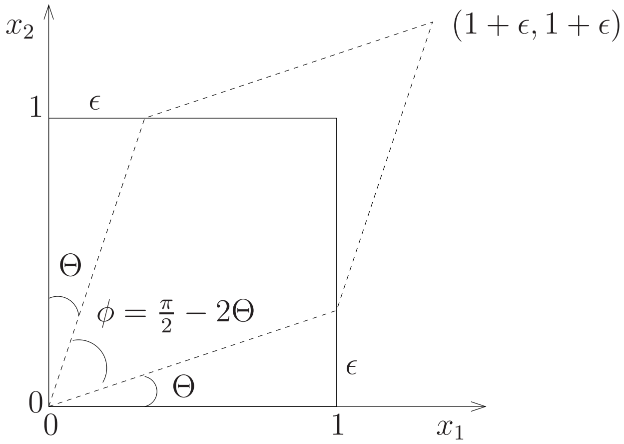


Figure 2.7: The undeformed unit square and its deformed configuration (dashed line) after application of the deformation $\boldsymbol{u} = (\epsilon x_2, \epsilon x_1)$.

which, as we would hope, agrees with the general theory.

Now let us turn to the second question. We can again use the general theory and formula (2.13) to deduce that the angle between deformed lines, ϕ, is given by

$$\cos\phi = 2e_{ij}n_i^{(1)}n_j^{(2)},$$

where $\boldsymbol{n}^{(1)}$ and $\boldsymbol{n}^{(2)}$ are unit vectors in the directions of the original lines. In this case $\boldsymbol{n}^{(1)} = (1, 0)$ and $\boldsymbol{n}^{(2)} = (0, 1)$ and we note that by symmetry of the strain tensor we will get the same answer if we had chosen $\boldsymbol{n}^{(1)} = (0, 1)$ and $\boldsymbol{n}^{(2)} = (1, 0)$. Hence,

$$\cos\phi = 2\,(1, 0)\begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\epsilon.$$

It follows that

$$\cos\phi = \sin(\pi/2 - \phi) \approx \pi/2 - \phi = 2\epsilon,$$

which is equal to the required change in angle. This can be seen directly from Figure 2.7, because the change in angle is 2Θ, but $\tan\Theta = \epsilon$ by using the right-angled triangles present. For small angles $\tan\Theta \approx \Theta = \epsilon$.

Finally, we address the third question. The principal axes of strain are found from solutions of the equation

$$(e_{ij} - \lambda\delta_{ij})\,v_j = 0.$$

The eigenvectors $\boldsymbol{v}$ cannot be trivial $\boldsymbol{v} \neq \boldsymbol{0}$ and non-trivial solutions are only possible if $\det(e_{ij} - \lambda\delta_{ij}) = 0$. In this case we require

$$\left|\begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = \begin{vmatrix} -\lambda & \epsilon \\ \epsilon & -\lambda \end{vmatrix} = 0.$$

Hence,

$$\lambda^2 - \epsilon^2 = 0 \quad \Rightarrow \quad \lambda = \pm\epsilon;$$

and the principal strains are $\lambda_1 = \epsilon$ and $\lambda_2 = -\epsilon$. Note that in two-dimensions because there are only two principal strains one must be the maximum and the other the minimum.

We now find the corresponding principal axes. If $\lambda = \epsilon$,

$$\begin{pmatrix} -\epsilon & \epsilon \\ \epsilon & -\epsilon \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Writing out each component gives

$$-v_1 + v_2 = 0, \quad \text{and} \quad v_1 - v_2 = 0.$$

As we should expect these equations are linearly dependent (because the matrix is singular by construction) and we have only one condition on the eigenvector $v_1 = v_2$. Thus the principal axis of strain is $(1, 1)$ with extension ratio ϵ.

If $\lambda = -\epsilon$, then

$$\begin{pmatrix} \epsilon & \epsilon \\ \epsilon & \epsilon \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Writing out each component gives

$$v_1 + v_2 = 0, \quad \text{and} \quad v_1 + v_2 = 0.$$

Again this is only one constraint, namely $v_1 = -v_2$ and the principal axis of strain is $(1, -1)$ with extension ratio $-\epsilon$ (a contraction).

Thus the maximum and minimum strains occur on the diagonals of the original square, which can be seen from the Figure 2.7.

2.5 Strain compatibility

We now know that the strain tensor is defined by Lecture 6

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and so given a displacement field u_i we can compute the strain tensor e_{ij} (assuming we can differentiate). The question now arises: given a strain field e_{ij} can we compute the corresponding displacement field u_i?

The strain tensor is symmetric so there are only *six* independent components and from the definition these are *six* partial differential equations (PDEs), but there are only *three* displacement components. Thus, in order to be able to recover the displacement there must be additional relationships between the e_{ij}.

Let us consider a (counter) example to illustrate the problem. Suppose that

$$e_{11} = x_2^2, \quad e_{12} = 0, \quad e_{22} = 0.$$

From the definition of the strain tensor

$$e_{11} = \frac{\partial u_1}{\partial x_1} = x_2^2 \quad \Rightarrow \quad u_1 = x_1 x_2^2 + f(x_2),$$

where $f(x_2)$ is an arbitrary function, rather than a constant because we are working with functions of many variables. Similarly,

$$e_{22} = \frac{\partial u_2}{\partial x_2} = 0 \quad \Rightarrow \quad u_2 = g(x_1),$$

where $g(x_1)$ is another arbitrary function. Now, let us consider the off-diagonal term

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0,$$

but from the expressions found above

$$\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2x_1x_2 + f'(x_2) + g'(x_1),$$

which cannot ever be identically zero no matter what form we choose for the functions f and g. Note that in the above the prime $'$ denotes differentiation with respect to the (only) argument of the function. If you are not convinced by the above argument then consider taking $\frac{\partial^2}{\partial x_1 \partial x_2}$ of the expression

$$\frac{\partial^2}{\partial x_1 \partial x_2}\left[2x_1x_2 + f'(x_2) + g'(x_1)\right] = 2 + 0 + 0 = 2 \neq 0.$$

However, if we now consider the slightly modified strain

$$e_{11} = x_2^2, \quad e_{12} = 0, \quad e_{22} = -x_1^2,$$

then

$$u_1 = x_1x_2^2 + f(x_2),$$

as before, but

$$u_2 = -x_2x_1^2 + g(x_1),$$

and then

$$e_{12} = \frac{1}{2}\left(2x_1x_2 + f'(x_2) - 2x_1x_2 + g'(x_1)\right),$$

and the troublesome "mixed" term disappears. The equation to be satisfied is now

$$f'(x_2) + g'(x_1) = 0,$$

which is only possible for all x_1 and x_2 if both functions are constants: $f'(x_2) = C$ and $g'(x_1) = -C$. On integration

$$f(x_2) = Cx_2 + U_1, \quad \text{and} \quad g(x_1) = -Cx_1 + U_2.$$

Thus, we have a displacement field

$$u_1 = x_1x_2^2 + Cx_2 + U_1, \quad \text{and} \quad u_2 = -x_2x_1^2 - Cx_1 + U_2,$$

and the unspecified constants define rigid-body motions: a rotation C and the translation (U_1, U_2). Thus, as we might expect, we can only recover the displacement field "up to" a rigid-body motion because the strain state is identical if two configurations differ only by a rigid-body motion.

Physical Picture

The strain tensor e_{ij} contains local information about how each small region of the continuum is deformed, but the pieces must still fit together if cracks or holes are not allowed (which they are not), see Figure 2.8. N.B. This is not true in fluids where the intermolecular bonds are weaker and regions originally in contact do not have to remain in contact.

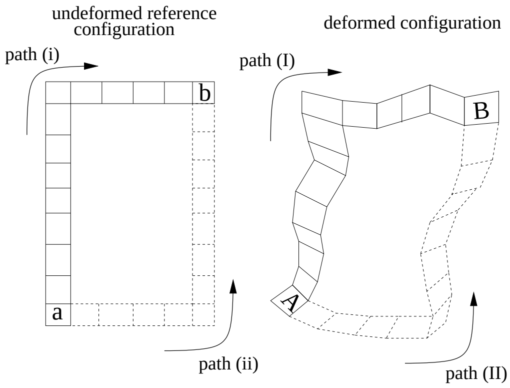


Figure 2.8: Sketch illustrating the strain compatibility condition.

The required equations of strain compatibility are given by

$$e_{ij,kl} + e_{kl,ij} - e_{kj,il} - e_{il,kj} = 0, \tag{2.16}$$

which actually represents $3^4 = 81$ different equations. Fortunately, both the strain tensor and the operation of partial differentiation are symmetric so many of the equations are trivial ($0 = 0$) or duplicates. In fact, there are only **six** non-trivial equations, represented in the table below.

i	1	1	1	1	1	2
j	1	1	2	1	2	2
k	2	2	2	3	3	3
l	2	3	3	3	3	3

In summary, if the strain compatibility equations (2.16) are satisfied then a *continuous* displacement field, $\boldsymbol{u}$, can be recovered, but $\boldsymbol{u}$ will not be unique because a rigid-body motion can still be added.

2.6 Homogeneous deformation

A deformation for which

$$\frac{\partial u_i}{\partial x_j} = \quad \text{const.} \tag{2.17}$$

throughout the body is called a *homogeneous deformation*. Consequently, the strain and rotation tensors are also constant and therefore the principal strains and axes of strain are the same throughout the body. Moreover, the strain compatibility equations are trivially satisfied because all partial derivatives of the strain tensor are zero.
Examples of homogeneous deformations:

Simple extension *e.g.* $e_{11} = e_0$, $e_{ij} = 0$ otherwise.

Uniform dilation $e_{ij} = e_0\delta_{ij}$, $d = e_{kk} = 3e_0$ (spherically symmetric).

Simple shear *e.g.* $e_{12} = e_{21} = e_0$, $e_{ij} = 0$ otherwise.

Chapter 3

Analysis of stress

3.1 Introductory Concepts

Lecture 7

When we consider forces acting on the surfaces of solid bodies then it was recognised long ago that rather than absolute force it is the force per unit area that is important. If a small area ΔS on the surface of a solid body is subject to a resultant force $\Delta\mathcal{F}$, then the traction (stress) vector $\mathbf{t}$ is defined to be:

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{\mathcal{F}}}{\Delta S} \tag{3.1}$$

and we assume that such a limit exists. The term 'traction' is usually used for stresses acting on the surfaces of a body.

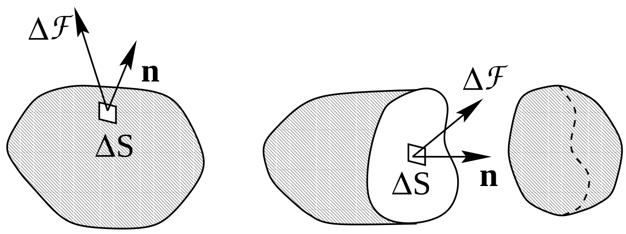


Figure 3.1: Sketch illustrating (external) traction and (internal) stress.

For internal forces there is no obvious surface, so we create one by pretending that we have cut the body in half along a plane with normal $\boldsymbol{n}$. We can then take an area ΔS in the cut plane and the resultant force $\Delta\mathcal{F}$ is the force exerted by the "other half" of the solid body at that point. We take the same limit as before to define

$$\boldsymbol{\tau} = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{\mathcal{F}}}{\Delta S}, \tag{3.2}$$

which represents the stress vector or "internal force per unit area". Note that $\boldsymbol{\tau}$ depends on the position within the body, $\boldsymbol{r}$, and the direction of the normal $\boldsymbol{n}$ to the cut plane (and by assumption nothing else!).

3.2 The stress tensor

We have just noted that the stress vector depends upon the normal vector $\boldsymbol{n}$, what is remarkable is that the relationship is linear, which means that we can write $\tau_i = \tau_{ij} n_j$, where τ_{ij} is called the stress tensor. In order to prove this we must consider the balance of forces within a small tetrahedral region, see Figure 3.2. Three of the tetrahedron's faces are in the planes x_i is constant and these

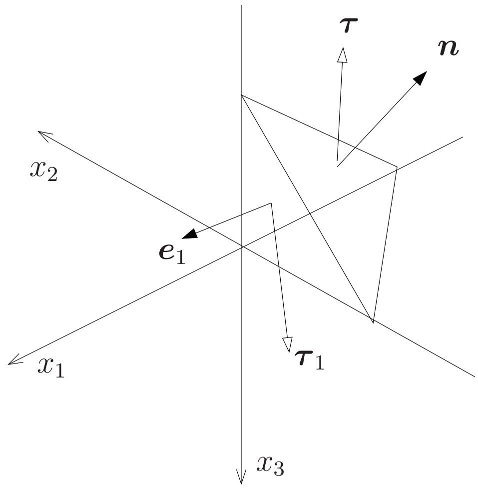


Figure 3.2: A tetrahedron is formed between Cartesian coordinate axes the coordinate x_i is a constant on face i, which has outer unit normal $\boldsymbol{e}_i$. The resultant stress vector acting on the i-th face is $\boldsymbol{\tau}_i$.

have area dS_i and outer unit normal $\boldsymbol{e}_i$. Thus we can represent these faces by a single vector with orientation $\boldsymbol{e}_i$ and magnitude equal to dS_i. The final face does not lie in any coordinate plane and have normal $\boldsymbol{n}$ and area dS. It follows from vector geometry that

$$\boldsymbol{e}_i dS_i + \boldsymbol{n} dS = 0, \qquad \text{(Exercise)}. \tag{3.3}$$

Taking the dot product of equation (3.3) with respect to $\boldsymbol{e}_j$ gives

$$\boldsymbol{e}_j \cdot \boldsymbol{e}_i dS_i + \boldsymbol{e}_j \cdot \boldsymbol{n} dS = 0,$$

and if we let $\boldsymbol{n} = (n_1, n_2, n_3)$ in the Cartesian basis then

$$\delta_{ij} dS_i + n_j dS = 0 \quad \Rightarrow \quad dS_j = -n_j dS. \tag{3.4}$$

We now assume that the resultant stress acting on the face on which x_i is constant is $\boldsymbol{\tau}_i$ and the resultant stress on the remaining face is $\boldsymbol{\tau}$. If the tetrahedron is in equilibrium then the total resultant force, the sum of the forces acting on all the faces, must be zero, *i.e.*,

$$\boldsymbol{\tau} dS + \boldsymbol{\tau}_j dS_j = \mathbf{0},$$

where we have used the fact that force is stress multiplied by area. [A bit of careful thought shows that for small tetrahedra body forces are negligible because the volume (cube of a small length) is much smaller than the the surface area (square of small length).] Thus,

$$\boldsymbol{\tau} dS = -\boldsymbol{\tau}_j dS_j = \boldsymbol{\tau}_j n_j dS,$$

after using equation (3.4). This demonstrates the required linear relationship between $\boldsymbol{\tau}$ and $\boldsymbol{n}$ and hence we must be able to write out in components

$$\tau_i = \tau_{ij} n_j,$$

as claimed above, where τ_{ij} is called the stress tensor. The component τ_{ij} represents the i-th component of the stress on the face whose outer unit normal points in the x_j direction, see Figure 3.3. Note that the components on opposite faces point in opposite directions because the sign of the normal is reversed.

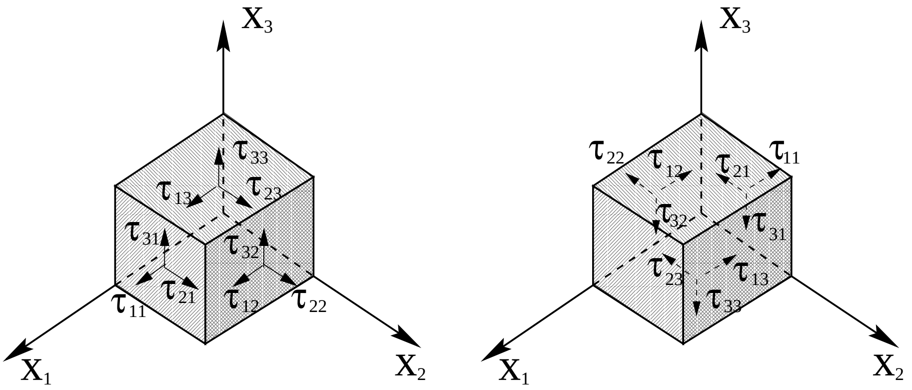


Figure 3.3: Sketch illustrating the components of the stress tensor.

We can now represent the internal stress state of the body in terms of the stress tensor, but it remains to determine τ_{ij} as a function of the state of strain.

3.3 Equilibrium of Forces

Consider an infinitesimal region of our elastic body that is loaded by an internal traction $\boldsymbol{\tau}$ on its surface and a body force $\boldsymbol{F}$ per unit volume. Hence, if the body is to be in equilibrium the total force must be zero:

$$\iint_S \boldsymbol{\tau}\,\mathrm{d}S + \iiint_V \boldsymbol{F}\,\mathrm{d}V = \boldsymbol{0}.$$

The above expression represents the sum of the surface (stress) and volume (body) forces. In component form, we can write

$$\iint_S \tau_i\,\mathrm{d}S + \iiint_V F_i\,\mathrm{d}V = \iint \tau_{ij} n_j\,\mathrm{d}S + \iiint_V F_i\,\mathrm{d}V = 0.$$

The surface integral can be converted into a volume integral by means of the divergence theorem,

$$\iiint_V \tau_{ij,j} + F_i\,\mathrm{d}V = 0.$$

The above integral must be zero for all possible volumes and so (assuming suitable existence and smoothness), we must have that

$$\tau_{ij,j} + F_i = 0 \quad \Rightarrow \quad \frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0, \tag{3.5}$$

which are the equations of equilibrium.

This equation can be generalised to the unsteady case via Newton's second law or, equivalently, D'Alembert's principle and we obtain the equations of motion

$$\rho \ddot{u}_i = \tau_{ij,j} + F_i, \tag{3.6}$$

where ρ is the density of the solid and $\ddot{\boldsymbol{u}} = \frac{\partial^2 \boldsymbol{u}}{\partial t^2}$ is the acceleration. These are sometimes called Cauchy's equations and are Newton's second law, $m\boldsymbol{a} = \boldsymbol{F}$, generalised to a solid.

3.4 Symmetry of the stress tensor and principal axes/stress invariants

The stress tensor is symmetric $\tau_{ij} = \tau_{ji}$ which can be proved by considering a moment (force $\times$ distance) balance about all axes parallel to $\boldsymbol{e}_i$ through the centre of the block. Consequently the discussion in section 2.4 applies to the stress tensor as well. Thus, the stress tensor has principal stresses, principal axes of stress and three invariants. In particular, we will denote the first invariant (the trace of the stress tensor) by

$$\Theta = \tau_{ii}. \tag{3.7}$$

Lecture 8

3.5 Homogeneous stress states

These states are analogous to homogeneous deformations (see section 2.6): Examples of homogeneous stress states:

Uniaxial stress *e.g.* $\tau_{33} = T_0$, $\tau_{ij} = 0$ otherwise.

Hydrostatic pressure $\tau_{ij} = p\delta_{ij}$ (spherically symmetric).

Pure shear stress *e.g.* $\tau_{12} = \tau_{21} = T_0$, $\tau_{ij} = 0$ otherwise.

3.6 Stress boundary conditions

There must be "continuity of stress" at all boundaries. Thus, if there is a given applied traction $\boldsymbol{t}$ at the surface of the elastic body, it must equal the internal stress at the surface:

$$\boldsymbol{\tau} = \boldsymbol{t} \quad \Rightarrow \quad \tau_i = t_i \quad \Rightarrow \quad \boxed{\tau_{ij} n_j = t_i}$$

where $\boldsymbol{n}$ is now the **outer** unit normal to the body.

Example

Consider a unit square loaded only by an external pressure, p_0 (no body forces), see Figure 3.4. Pressure is **always** directed normally into the body. Thus, for any face, a pressure load is $\boldsymbol{t} = -p_0\boldsymbol{n}$ if $\boldsymbol{n}$ is the outer unit normal.

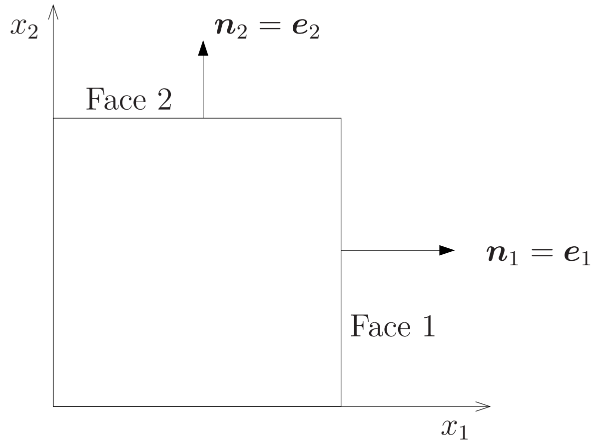


Figure 3.4: A unit square is loaded only by an external pressure p_0.

On Face 1, the load is $\boldsymbol{t} = -p_0\boldsymbol{e}_1 = -p_0\boldsymbol{n}_1$. Thus, $t_1 = -p_0$ and $t_2 = 0$ because $\boldsymbol{n} = (1, 0)$. From continuity of stress $\tau_{ij}n_j = t_i$, so putting $i = 1$ and expanding the sum yields

$$\tau_{11}n_1 + \tau_{12}n_2 = t_1 = -p_0 \quad \Rightarrow \quad \tau_{11} = -p_0.$$

Note that τ_{11} represents the stress component in the positive x_1 direction. Hence it is negative in this case because the body is compressed: the traction (pressure) acts into the body. Repeating the same procedure for $i = 2$ we obtain

$$\tau_{21}n_1 + \tau_{22}n_2 = t_2 = 0 \quad \Rightarrow \quad \tau_{21} = \tau_{12} = 0.$$

Note that we do not get boundary conditions for all components of the stress tensor!

On Face 2, the load is $\boldsymbol{t} = -p_0\boldsymbol{e}_2 = -p_0\boldsymbol{n}_2$. Thus, $t_1 = 0$ and $t_2 = -p_0$ because $\boldsymbol{n}_2 = (0, 1)$. From continuity of stress we have

$$i = 1: \qquad \tau_{11}n_1 + \tau_{12}n_2 = \tau_{12} = \tau_{21} = 0,$$

$$i = 2: \qquad \tau_{21}n_1 + \tau_{22}n_2 = \tau_{22} = -p_0.$$

Thus, if the stress tensor takes the form $\tau_{ij} = -p_0\delta_{ij}$ it satisfies all stress boundary conditions on both faces; and we see that a pure pressure loading yields a hydrostatic pressure stress tensor. Note that this trivially satisfies the equations of equilibrium because it is a constant.

3.7 Resultant force and moments

For a given stress distribution, $\boldsymbol{\tau}$ at the surface of a body the resultant force is the integral of the stress over the body's surface:

$$\boldsymbol{F} = \iint_S \boldsymbol{\tau}\,\mathrm{d}S \quad \text{or} \quad F_i = \iint_S \tau_{ij}n_j\,\mathrm{d}S.$$

The resultant moment about a given axis is

$$\boldsymbol{M} = \iint_S \boldsymbol{r} \times \boldsymbol{\tau}\,\mathrm{d}S,$$

where $\boldsymbol{r}$ is the vector distance from the axis to a point on the surface and $\times$ is the usual vector cross product.

Chapter 4

Elasticity & constitutive equations

4.1 The constitutive equations

Lecture 9

Thus far we have characterized deformation by strains (see chapter 2) and internal forces by stresses (see chapter 3), from which we could deduce equations of motion. We now ask the obvious question:

How do the stresses relate to the strains?

The relationship between stresses and strains is called the constitutive law and characterises the behaviour of the material under consideration.

In a general *elastic* body, the stresses are functions only of the instantaneous, local strains:

$$\tau_{ij}(\boldsymbol{r}, t) = \tau_{ij}(e_{kl}(\boldsymbol{r}, t)), \tag{4.1}$$

and each of the six independent components of the stress tensor can depend on all six independent components of the strain tensor.

For *small* strains, a Taylor expansion of (4.1) gives

$$\tau_{ij} = \underbrace{\tau_{ij}\big|_{e_{kl}=0}}_{\text{Initial Stress } \tau_{ij}^{0}} + \underbrace{\left.\frac{\partial \tau_{ij}}{\partial e_{kl}}\right|_{e_{kl}=0}}_{E_{ijkl}} e_{kl}, \tag{4.2}$$

which motivates the definition that a solid is linearly elastic if:

$$\tau_{ij} = \tau_{ij}^{(0)} + E_{ijkl} e_{kl}. \tag{4.3}$$

If the initial stress in the undeformed configuration is zero, then we obtain Hooke's law:

$$\tau_{ij} = E_{ijkl} e_{kl}. \tag{4.4}$$

Most solids are linearly elastic for sufficiently small strains.

The 4-th rank tensor E_{ijkl} has $3^4 = 81$ coefficients, but because both e_{ij} and τ_{ij} are symmetric the coefficients of E_{ijkl} are not all independent. In fact, there are only 21 independent coefficients, but these must be determined from experiments.

Definition: A solid body is called *homogeneous* if E_{ijkl} is independent of x_i.

Definition: A solid body is called *isotropic* if its elastic properties are the same in all directions; (*i.e.* no reinforcing fibres or other "directed" internal structures).

For an isotropic, homogeneous, elastic solid there are only two independent coefficients that must be determined from experiment and

$$E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl}. \tag{4.5}$$

Here, λ and μ are the independent coefficients known as the *Lamé constants.*

Hence, the constitutive law for an isotropic, homogeneous, linearly elastic solid is

$$\tau_{ij} = E_{ijkl}e_{kl} = (\lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl})\, e_{kl},$$

$$\Rightarrow \quad \tau_{ij} = \lambda\delta_{ij} \underbrace{e_{kk}}_{=d} + 2\mu e_{ij}. \tag{4.6}$$

Note that the principal axes of e_{ij} and τ_{ij} coincide (exercise). Also if there is no volume change then $d = 0$ and τ_{ij} is linearly proportional to e_{ij}.

Inverse Form

Equation (4.6) gives an explicit expression for the stress in terms of the strain, but what is the strain for a given stress? If $d = 0$, the answer is simple, but in general we need to work a little harder. Firstly, set $i = j$ in equation (4.6)

$$\tau_{ii} = 3\lambda e_{kk} + 2\mu e_{ii} = (3\lambda + 2\mu)\, d \quad \Rightarrow \quad \Theta = (3\lambda + 2\mu)\, d,$$

where $\Theta = \tau_{ii}$ and $d = e_{ii}$.

Rearranging (4.6) gives

$$e_{ij} = \frac{1}{2\mu}\left(\tau_{ij} - \lambda\delta_{ij} d\right),$$

$$\Rightarrow \quad e_{ij} = \frac{1}{2\mu}\left(\tau_{ij} - \lambda\delta_{ij}\frac{\Theta}{3\lambda + 2\mu}\right).$$

Thus,

$$e_{ij} = \frac{1}{2\mu}\tau_{ij} - \frac{\lambda}{2\mu\,(3\lambda + 2\mu)}\delta_{ij}\tau_{kk},$$

which can be written as

$$e_{ij} = D_{ijkl}\tau_{kl}, \tag{4.7}$$

where

$$D_{ijkl} = \frac{1}{2\mu}\delta_{ik}\delta_{jl} - \frac{\lambda}{2\mu\,(3\lambda + 2\mu)}\delta_{ij}\delta_{kl}.$$

Rest of chapter is reading week material and not covered explicitly in lectures

4.2 Experimental determination of elastic constants

4.2.1 Experiment I: Simple extension of a thin cylinder

A circular cylinder of diameter D, cross-sectional area A and initial length L is subject to an applied tension T along its axis which is chosen to lie in the x_3 direction. After the load has been applied the cylinder has length $L + \Delta L$ and diameter $D + \Delta D$.

The extension is in the x_3 direction and so $e_{33} = \Delta L/L \ll 1$. The change in diameter is in the x_1 and x_2 directions, so $e_{11} = e_{22} = \Delta D/D$. Moreover, the stress is uniaxial (along one axis, x_3) so $\tau_{33} = T/A$, and $\tau_{ij} = 0$ otherwise.

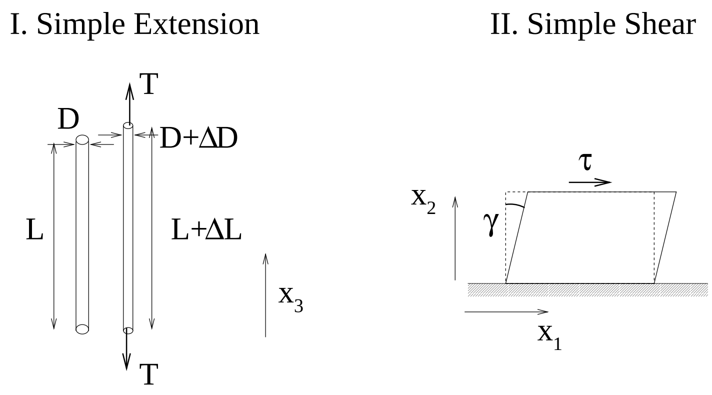


Figure 4.1: Sketch illustrating the two fundamental experiments for the determination of the elastic constants.

- Observations:
  - Rod extends in the x_3 direction in proportion to the applied tension:

$$\tau_{33} = E e_{33}, \quad \text{where } E \text{ is the Elastic (or Young's) modulus.} \tag{4.8}$$

  - Rod cross-section decreases (contracts in plane) in proportion to its extension in the axial direction:

$$e_{11} = e_{22} = -\nu\, e_{33}, \quad \text{where } \nu \text{ is the Poisson ratio.} \tag{4.9}$$

4.2.2 Experiment II: Simple shear

A block in the x_1-x_2 plane is subject to an applied traction, τ, on its upper surface in the positive x_1 direction that induces a change in angle $\gamma = 2e_{12}$ between two sides. The body is in a state of simple shear so $\tau_{12} = \tau$ and $\tau_{ij} = 0$ otherwise.

- Observation:
  - The change in angle is proportional to the applied traction:

$$\tau = G\gamma \Rightarrow \tau_{12} = G\; 2e_{12}, \quad \text{where } G \text{ is the shear modulus of the material.} \tag{4.10}$$

4.3 Relating experiments (E, ν, G) and theory (λ, μ)

We shall now use the constitutive equations to relate the experimentally measured constants E, ν and G to the (theoretical) Lamé coefficients λ and μ. The material is homogeneous, isotropic and elastic so that

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu\, e_{ij}, \tag{4.11}$$

and, alternatively,

$$e_{ij} = \frac{1}{2\mu}\tau_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij}\tau_{kk}. \tag{4.12}$$

Experiment I

In this experiment, $\tau_{33} = Ee_{33}$ and $\Theta = \tau_{kk} = \tau_{33}$, because $\tau_{11} = \tau_{22} = 0$. If we set $i = j = 3$ in equation (4.12) we have

$$e_{33} = \frac{1}{2\mu}\tau_{33} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\tau_{kk} = \left[\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\right]\tau_{33} = \left[\frac{3\lambda+2\mu-\lambda}{2\mu(3\lambda+2\mu)}\right]\tau_{33} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\tau_{33}.$$

Hence,

$$\frac{1}{E} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \quad \Rightarrow \quad \boxed{E = \tfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}.} \tag{4.13}$$

If we substitute $i = j = 1$ into equation (4.12), we obtain

$$e_{11} = \frac{1}{2\mu}\tau_{11} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\tau_{33} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\tau_{33}, \qquad \text{because } \tau_{11} = 0.$$

Using the first experimental observation, $\tau_{33} = Ee_{33}$, and the expression (4.13) for E gives

$$e_{11} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}e_{33} = -\frac{\lambda}{2(\lambda+\mu)}\,e_{33}.$$

The second experimental observation is $e_{11} = -\nu\, e_{33}$, so direct comparison yields

$$\boxed{\nu = \tfrac{\lambda}{2(\lambda+\mu)}.} \tag{4.14}$$

Note that an identical result is obtained on substitution of $i = j = 2$ into equation (4.12).

Experiment II

In this experiment, $\tau_{12} = 2Ge_{12}$, and using equation (4.11) with $i = 1$ and $j = 2$, we have

$$\tau_{12} = \lambda\delta_{12}e_{kk} + 2\mu\, e_{12} = 2\mu, e_{12} \quad \Rightarrow \quad \boxed{\mu = G.} \tag{4.15}$$

4.4 Relations between the elastic constants

Note that only two elastic constants are independent, see equations (4.11, 4.12). The following table lists the relationships between elastic coefficients for the most commonly used independent pairs.

Independent pair	$\lambda =$	$\mu = G =$	$E =$	$\nu =$
λ, μ	λ	μ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$
λ, ν	λ	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{(1+\nu)(1-2\nu)\lambda}{\nu}$	ν
μ, E	$\frac{\mu(E-2\mu)}{3\mu-E}$	μ	E	$\frac{E-2\mu}{2\mu}$
E, ν	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	E	ν

4.4.1 Constitutive equations in terms of the experimentally measurable constants

Using the table above, we can rewrite the equation (4.11) and (4.12) using the Young's modulus E and Poisson ratio ν

$$\tau_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\underbrace{e_{kk}}_{d}\right). \tag{4.16}$$

and

$$e_{ij} = \frac{1}{E}\left((1+\nu)\tau_{ij} - \nu\,\delta_{ij}\underbrace{\tau_{kk}}_{\Theta}\right). \tag{4.17}$$

For physically realisable materials,

$$E \geq 0, \quad -1 < \nu \leq \frac{1}{2}, \quad \mu \geq 0, \quad 3\lambda + 2\mu > 0.$$

Note that equation (4.17) implies that

$$d = e_{kk} = \frac{\tau_{kk}}{E}\left((1+\nu) - 3\nu\right) = \frac{\Theta}{E}\left(1-2\nu\right),$$

and so $d = 0$, when $\nu = 1/2$. Materials for which $\nu = 1/2$ are thus said to be *incompressible*. Care must be taken when considering the limit $\nu \to 1/2$ in equation (4.16). In fact, the distinguished limit in question is $\nu \to 1/2$ **and** $d \to 0$, in such a way that the term $d/(1-2\nu)$ remains finite.

Chapter 5

The governing equations of linear elasticity

5.1 Navier–Lamé equations — displacement formulation

Lecture 10

In the previous chapters we have established the following equations:

$$\text{Definition of strain} \qquad e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{5.1a}$$

$$\text{Equations of motion} \qquad \tau_{ij,j} + F_i = \rho \tfrac{\partial^2 u_i}{\partial t^2}, \tag{5.1b}$$

$$\text{Linear elastic constitutive law} \qquad \tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \tag{5.1c}$$

The idea is to eliminate the stress and strain to obtain an equation that only contains the displacement $\boldsymbol{u}$. Using equation (5.1a) in equation (5.1c) we obtain an expression for the stresses in terms of the displacements

$$\tau_{ij} = \lambda\, \delta_{ij}\, u_{k,k} + \mu\, (u_{i,j} + u_{j,i})\,. \tag{5.2}$$

Now, taking the derivative of equation (5.2) gives

$$\tau_{ij,j} = \lambda\, \delta_{ij}\, u_{k,kj} + \mu\, (u_{i,jj} + u_{j,ij}) = \lambda\, u_{k,ki} + \mu\, u_{i,jj} + \mu u_{j,ji} = (\lambda + \mu) u_{k,ki} + \mu\, u_{i,jj},$$

by using the symmetry of the partial derivative and the index-switching property of the Kronecker delta. Using the last result in equation (5.1b), we obtain the Navier–Lamé equations

$$\boxed{\rho \ddot{u}_i = (\lambda + \mu)\, u_{k,ki} + \mu\, u_{i,jj} + F_i.} \tag{5.3}$$

The Navier–Lamé equations may be written in the equivalent form

$$\rho \ddot{\boldsymbol{u}} = (\lambda + \mu)\, \mathrm{grad}(\mathrm{div}\, \boldsymbol{u}) + \mu\, \nabla^2 \boldsymbol{u} + \boldsymbol{F},$$

or even

$$\rho \ddot{\boldsymbol{u}} = (\lambda + 2\mu)\, \mathrm{grad}(\mathrm{div}\, \boldsymbol{u}) - \mu\, \mathrm{curl}(\mathrm{curl}\, \boldsymbol{u}) + \boldsymbol{F},$$

by using the vector identity

$$\nabla^2 \boldsymbol{u} = \mathrm{grad}(\mathrm{div}\, \boldsymbol{u}) - \mathrm{curl}(\mathrm{curl}\, \boldsymbol{u}).$$

However they are written these Navier–Lamé equations are a system of three coupled linear elliptic PDEs for the three displacements $u_i(x_j)$.

Note

- The general case of a non-zero body force is not very different from the "special" case $\boldsymbol{F} = \boldsymbol{0}$ because the Navier–Lamé equations are linear. $\boldsymbol{F}$ acts as an inhomogeneity that can be removed if a suitable particular solution is found. We seek a function $\hat{\boldsymbol{u}}$ such that

$$(\lambda + \mu)\, \boldsymbol{\nabla\nabla} \cdot \hat{\boldsymbol{u}} + \mu\, \nabla^2 \hat{\boldsymbol{u}} + \boldsymbol{F} = 0,$$

irrespective of the boundary conditions. If we then define the displacement field to be $\boldsymbol{u} = \tilde{\boldsymbol{u}} + \hat{\boldsymbol{u}}$, then $\tilde{\boldsymbol{u}}$ satisfies the homogeneous equation

$$(\lambda + \mu)\, \boldsymbol{\nabla\nabla} \cdot \tilde{\boldsymbol{u}} + \mu\, \nabla^2 \tilde{\boldsymbol{u}} = 0,$$

but the boundary conditions may have changed from the original problem.

5.2 Boundary conditions for the Navier–Lamé equations:

As always in mathematical modelling, the equation is only part of the story. We must also consider the boundary conditions that should be imposed. Some physically sensible boundary conditions for the Navier–Lamé equations are illustrated in Figure 5.1.

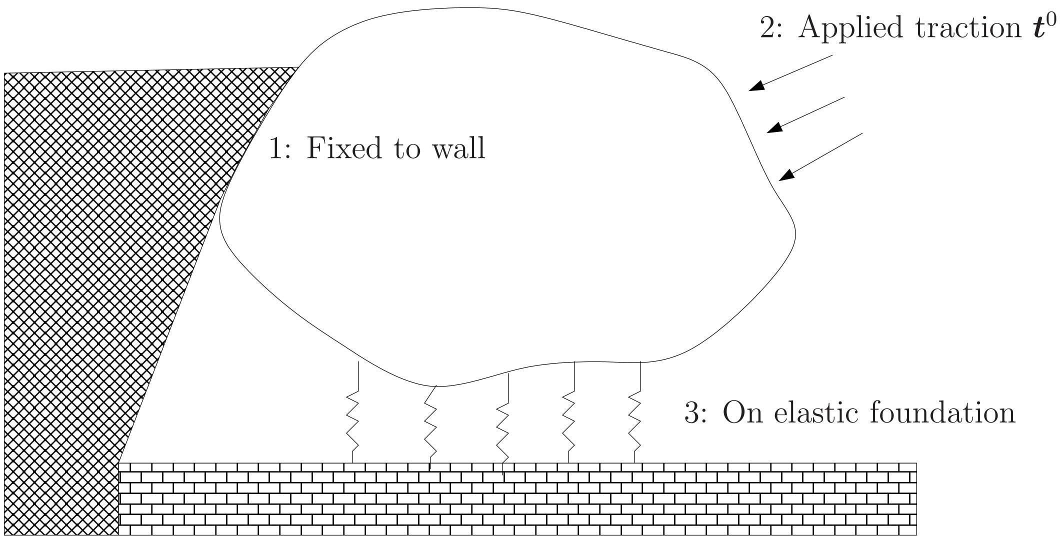


Figure 5.1: Physically sensible boundary conditions that could be applied to a solid body.

1. Fixed or prescribed displacement field, $\boldsymbol{u}^{(0)}$.
   The boundary condition is simply
$$u_i = u_i^{(0)},$$
   on the boundary, which corresponds to a Dirichlet condition for $\boldsymbol{u}$.

2. Prescribed traction, $\boldsymbol{t}^{(0)}$.
   Prescribing the traction is actually a boundary condition on the stress and from continuity of stress
$$t_i^{(0)} = \tau_{ij} n_j \quad \Rightarrow \quad t_i^{(0)} = \left(\lambda \delta_{ij} u_{k,k} + \mu(u_{i,j} + u_{j,i})\right) n_j,$$

from the constitutive law. The resulting equation is a condition on the derivatives of $\boldsymbol{u}$ at the boundary, which is a Neumann boundary condition.

3. Elastic bedding.
   In this case the applied traction is related to the displacement of the boundary. Consider the simplified problem in which there is a single linear spring connected to our elastic body that can move only in the x_1 direction, see Figure 5.2.

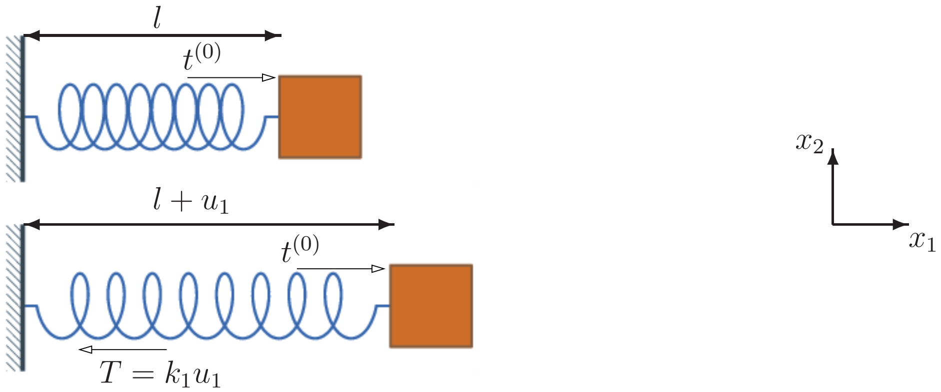


Figure 5.2: An elastic body is attached by spring of undeformed length l to a solid wall. The elastic body is subject to the external traction $t^{(0)}$ in the positive x_1 direction (upper image). After application of the traction the spring extended to the length $l + u_1$, where u_1 is the displacement of the solid body in the x_1 direction (lower image). For a linear spring, the tension (restoring force) is $T = k_1 u_1$ which acts in the negative x_1 direction.

If we assume that the spring is in its rest state when $u_1 = 0$ then the tension in the spring is $T = k_1 u_1$, where k_1 is the spring constant and it acts in the negative x_1 direction. Resolving forces in the x_1 direction gives

$$\underbrace{t_1^{(0)}}_{\text{Applied traction}} - \underbrace{k_1 u_1}_{\text{Restoring force}} = \tau_{1j} n_j = \underbrace{-\tau_{11}}_{\text{stress at surface}} .$$

Thus, the surface stress is related to the displacement through the restoring force applied by the spring.

For a general (linear) elastic foundation

$$t_i^{(0)} = \tau_{ij} n_j + k_{ij} u_j,$$

where k_{ij} is a stiffness tensor (or matrix of spring coefficients). This type of boundary condition is a mixed or Robin condition.

Whatever boundary conditions are applied, once we have solved the Navier–Lamé equations (5.3) to find the displacement field $\boldsymbol{u}$, we can use equations (5.1a) and (5.1c) to determine the corresponding strain and stress fields.

5.3 Beltrami–Michell equations — stress formulation

The Navier–Lamé equations treat the displacements as the unknowns in elasticity problems; the strains and stresses are calculated indirectly, once the displacements are known. An alternative

Remainder of chapter is reading week material and not covered explicitly in lectures approach is to formulate the set of governing equations so that the stresses are the unknowns and are calculated directly. We can then use the constitutive law to determine the strains and hence the displacements. There is a potential problem, however, the displacements cannot be uniquely recovered from the strain field unless the strain compatibility conditions are satisfied.

$$e_{ij,kl} + e_{kl,ij} - e_{kj,il} - e_{il,kj} = 0. \tag{5.4}$$

The inverse form of the linear constitutive law (5.1c) is

$$e_{ij} = \frac{1}{2\mu}\tau_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\tau_{kk},$$

and using the engineering (experimental) constants E and ν, we obtain a slightly neater expression

$$\frac{E}{1+\nu}e_{ij} = \tau_{ij} - \frac{\nu}{1+\nu}\delta_{ij}\Theta. \tag{5.5}$$

Substituting the expression (5.5) into the strain compatibility equations (5.4) yields

$$\tau_{ij,kl} + \tau_{kl,ij} - \tau_{kj,il} - \tau_{il,kj} - \frac{\nu}{1+\nu}\Big[\delta_{ij}\Theta_{,kl} + \delta_{kl}\Theta_{,ij} - \delta_{kj}\Theta_{,il} - \delta_{il}\Theta_{,kj}\Big] = 0.$$

All the terms in the above expression are second derivatives of τ_{ij}, so that the equations are all satisfied if the stress is a **linear** function of position, in which case every term is zero. Remember that only six of these 81 equations are linearly independent, so we are free to combine equations by setting $k = l$ and summing over the repeated index to obtain nine equations, of which six are still independent through the symmetry of the stress tensor,

$$\tau_{ij,kk} + \tau_{kk,ij} - \tau_{kj,ik} - \tau_{ik,kj} - \frac{\nu}{1+\nu}\left[\delta_{ij}\Theta_{,kk} + \delta_{kk}\Theta_{,ij} - \delta_{kj}\Theta_{,ik} - \delta_{ik}\Theta_{,kj}\right] = 0.$$

Using the fact that $\tau_{kk} = \Theta$, $\delta_{kk} = 3$ and the index-switching property of the Kronecker delta, the equations become

$$\tau_{ij,kk} + \Theta_{,ij} - \tau_{kj,ik} - \tau_{ik,kj} - \frac{\nu}{1+\nu}\left[\delta_{ij}\Theta_{,kk} + 3\Theta_{,ij} - \Theta_{,ij} - \Theta_{,ij}\right] = 0,$$

$$\Rightarrow \tau_{ij,kk} + \Theta_{,ij} - \tau_{kj,ik} - \tau_{ik,kj} - \frac{\nu}{1+\nu}\left[\delta_{ij}\Theta_{,kk} + \Theta_{,ij}\right] = 0,$$

$$\Rightarrow \tau_{ij,kk} + \frac{1}{1+\nu}\Theta_{,ij} - \frac{\nu}{1+\nu}\Theta_{,kk}\delta_{ij} - \tau_{kj,ik} - \tau_{ik,kj} = 0. \tag{5.6}$$

The equation (5.6) can be written in the alternative form

$$\nabla^2\tau_{ij} + \frac{1}{1+\nu}\Theta_{,ij} - \frac{\nu}{1+\nu}\delta_{ij}\nabla^2\Theta = \tau_{kj,ik} + \tau_{ik,kj}. \tag{5.7}$$

Differentiating the equations of static equilibrium $\tau_{ij,j} + F_i = 0$, we obtain $\tau_{ij,jk} + F_{i,k} = 0$, which can be used to replace the terms on the right-hand side of equation (5.7).

$$\nabla^2\tau_{ij} + \frac{1}{1+\nu}\Theta_{,ij} - \frac{\nu}{1+\nu}\delta_{ij}\nabla^2\Theta = -F_{j,i} - F_{i,j}. \tag{5.8}$$

Setting $i = j$ in equation (5.8) gives

$$\nabla^2\tau_{ii} + \frac{1}{1+\nu}\Theta_{,ii} - \frac{\nu}{1+\nu}3\nabla^2\Theta = -2F_{i,i} \quad \Rightarrow \nabla^2\Theta + \frac{1}{1+\nu}\nabla^2\Theta - \frac{3\nu}{1+\nu}\nabla^2\Theta = -2F_{i,i}$$

$$\Rightarrow \frac{1-\nu}{1+\nu}\nabla^2\Theta = -F_{i,i}; \tag{5.9}$$

and on substitution of (5.9) into equation (5.8) we finally obtain the Beltrami–Michell (compatibility) equations

$$\boxed{\nabla^2\tau_{ij} + \tfrac{1}{1+\nu}\,\Theta_{,ij} + \tfrac{\nu}{1-\nu}\,\delta_{ij}F_{k,k} + F_{i,j} + F_{j,i} = 0.} \tag{5.10}$$

These are stress compatibility equations and we note that in unsteady problems, we must add the inertia force $-\rho\ddot{u}_i$ to F_i.

The Beltrami–Michell equations are six linear partial differential equations for the six independent components of the stress tensor. As one might expect, these equations are (very) useful if the problem has stress boundary conditions. The equations are less useful if displacement boundary conditions are specified.

Notes

- The equations of equilibrium are automatically satisfied by construction.
- The strains e_{ij} can be recovered from the constitutive equation.
- Once the strain field is known, the displacements can be obtained by integrating

  $$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right).$$

  Remember that the constants of integration represent rigid-body motions.
- The above integration is possible because we started from the strain compatibility equations and, therefore, if the Beltrami–Michell equations are satisfied so are the strain compatibility equations; and hence, the body will be continuous.

5.4 The steady, constant-body-force case

If the body force is a constant and there are no unsteady terms, then all of the derivatives $F_{i,j}$ will be zero. The Beltrami–Michell equations become

$$\nabla^2\tau_{ij} + \frac{1}{1+\nu}\Theta_{,ij} = 0,$$

and from equation (5.9) we have

$$\nabla^2\Theta = 0.$$

Thus, Θ is a harmonic function: a function that satisfies Laplace's equation. Recall that $\Theta = (3\lambda + 2\mu)\,d$, which implies that

$$\nabla^2 d = 0,$$

and so $d = e_{kk} = u_{k,k}$ is also a harmonic function. If we apply the operator ∇^2 to the (steady) Navier–Lamé equations (5.3) we obtain

$$(\lambda + \mu)\,u_{k,kijj} + \mu\,u_{i,kkjj} = 0.$$

The first term may be written as $(u_{k,kjj})_{,i}$ and because $u_{k,k}$ is harmonic then $u_{k,kjj} = 0$ and hence the first term is zero. Thus,

$$u_{i,kkjj} = 0, \quad \text{or rather} \quad \boxed{\nabla^4 \boldsymbol{u} = 0,}$$

and so the displacement field is a solution of the biharmonic equation. Furthermore, by using the definition of the strain tensor (5.1a) and the expression for the stress tensor in terms of the displacements (5.2), it follows that

$$\boxed{\nabla^4 \tau_{ij} = 0, \quad \text{and} \quad \nabla^4 e_{ij} = 0;}$$

the stress and strain components are both biharmonic functions.

5.5 Alternative coordinate systems:

5.5.1 Governing Equations in Cylindrical Polar Coordinates

- $x_1 = x = r\cos\theta$, $x_2 = y = r\sin\theta$, $x_3 = z = z$.

$$\mathbf{u} = (u_r, u_\theta, u_z), \quad e = (e_{ij}), \quad \tau = (\tau_{ij}), \quad \text{where } i, j = r, \theta, z.$$

- Vector calculus:

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl}\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

- Stress-strain relations have the same form as in Cartesian coordinates:

$$\tau_{ij} = \lambda\delta_{ij}\operatorname{div}\mathbf{u} + 2\mu e_{ij}, \qquad i, j = r, \theta, z.$$

- Stress-displacement relations:

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda\operatorname{div}\mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Strain-displacement relations:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{r\theta} = 2e_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad 2e_{rz} = 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad 2e_{z\theta} = 2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Equilibrium equations (statics): for the displacement formulation, use Navier's equation,

$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{F} = \mathbf{0},$$

  whereas for the stress formulation, use

$$\begin{aligned}
\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + F_r &= 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + F_\theta &= 0 \\
\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} + F_z &= 0.
\end{aligned}$$

- Stress boundary conditions: these are when $\mathbf{t}$ is prescribed. We have, from $t_i = \hat{n}_j\tau_{ij}$,

$$\begin{aligned}
t_r &= \hat{n}_r\tau_{rr} + \hat{n}_\theta\tau_{r\theta} + \hat{n}_z\tau_{rz} \\
t_\theta &= \hat{n}_r\tau_{r\theta} + \hat{n}_\theta\tau_{\theta\theta} + \hat{n}_z\tau_{\theta z} \\
t_z &= \hat{n}_r\tau_{rz} + \hat{n}_\theta\tau_{\theta z} + \hat{n}_z\tau_{zz}
\end{aligned}$$

5.5.2 Governing Equations in Spherical Polar Coordinates

- $x_1 = x = r\sin\theta\,\cos\phi$, $x_2 = y = r\sin\theta\,\sin\phi$, $x_3 = z = r\cos\theta$.

$$\mathbf{u} = (u_r, u_\theta, u_\phi), \quad e = (e_{ij}), \quad \tau = (\tau_{ij}), \quad \text{where } i, j = r, \theta, \phi.$$

- Vector calculus:

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}},$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\,u_r) + \frac{\partial}{\partial\theta}(r\sin\theta\,u_\theta) + \frac{\partial}{\partial\phi}(ru_\phi)\right\},$$

$$\operatorname{curl}\mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ u_r & ru_\theta & r\sin\phi\,u_\phi \end{vmatrix}.$$

- Stress-strain relations have the same form as in Cartesian coordinates:

$$\tau_{ij} = \lambda\delta_{ij}\operatorname{div}\mathbf{u} + 2\mu e_{ij}, \qquad i, j = r, \theta, \phi.$$

- Stress-displacement relations:

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right),$$

$$\tau_{\phi\phi} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi} + u_r + u_\theta\cot\theta\right), \quad \frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta},$$

$$\frac{\tau_{r\phi}}{\mu} = \frac{\tau_{\phi r}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}, \quad \frac{\tau_{\theta\phi}}{\mu} = \frac{\tau_{\phi\theta}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi\cot\theta}{r}.$$

- Strain-displacement relations:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \qquad e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r},$$

$$2e_{r\theta} = 2e_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}, \quad 2e_{r\phi} = 2e_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r},$$

$$2e_{\phi\theta} = 2e_{\theta\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi\cot\theta}{r}.$$

- Equilibrium equations (statics): for the displacement formulation, use Navier's equation,

$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{F} = \mathbf{0},$$

whereas for the stress formulation, use

$$\begin{aligned} \frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} + \frac{2\tau_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi} + \cot\theta\,\tau_{r\theta}}{r} + F_r &= 0 \\ \frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{3\tau_{r\theta} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta}{r} + F_\theta &= 0 \\ \frac{\partial\tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{3\tau_{r\phi} + 2\tau_{\theta\phi}\cot\theta}{r} + F_\phi &= 0. \end{aligned}$$

- Stress boundary conditions: these are when $\mathbf{t}$ is prescribed. We have, from $t_i = \hat{n}_j \tau_{ij}$,

$$\begin{aligned} t_r &= \hat{n}_r \tau_{rr} + \hat{n}_\theta \tau_{r\theta} + \hat{n}_\phi \tau_{r\phi} \\ t_\theta &= \hat{n}_r \tau_{r\theta} + \hat{n}_\theta \tau_{\theta\theta} + \hat{n}_\phi \tau_{\theta\phi} \\ t_\phi &= \hat{n}_r \tau_{r\phi} + \hat{n}_\theta \tau_{\theta\phi} + \hat{n}_\phi \tau_{\phi\phi} \end{aligned}$$

Chapter 6

Problems with variation in only one direction (1D)

6.1 Infinite cylindrical pipe under internal pressure

Lecture 11

An infinite cylindrical pipe is loaded by an internal pressure p_0, see Figure 6.1, and has inner radius a and b. The pipe is made of an isotropic, linearly elastic material with Lamé constants λ and μ.

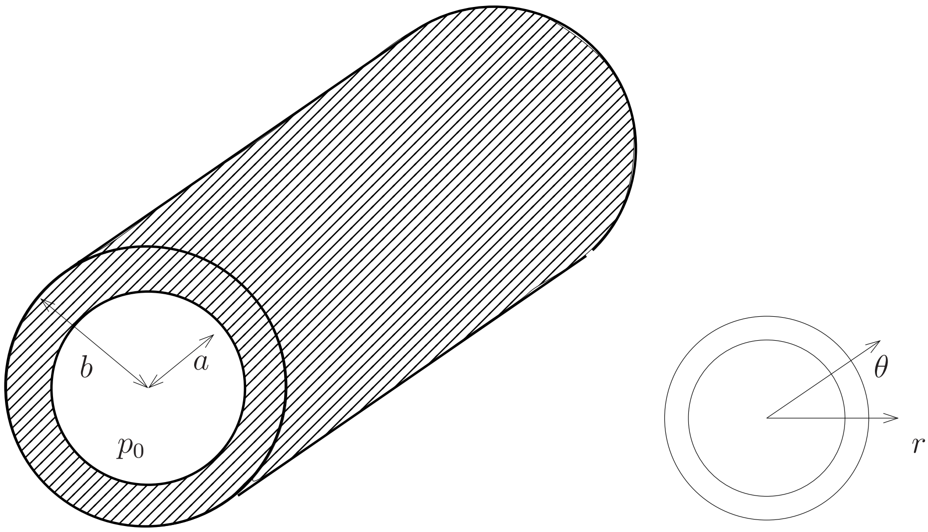


Figure 6.1: An infinite elastic cylindrical pipe has inner radius a and outer radius b. It is loaded internally by a pressure p_0.

We wish to find the displacement field within the pipe and then find an explicit expression for the "hoop stress", $\tau_{\theta\theta}$. Note that there are no body forces acting and that any "missing" boundary conditions (those that are not mentioned explicitly) should be treated as zero traction, which means that the boundary is not loaded.

It is most natural to use cylindrical polar coordinates (r, θ, z) for this problem, in which case

$$\boldsymbol{u}(r, \theta, z) = u_r(r, \theta, z)\, \boldsymbol{e}_r + u_\theta(r, \theta, z)\, \boldsymbol{e}_\theta + u_z(r, \theta, z)\, \boldsymbol{e}_z.$$

Let us now consider the boundary conditions on each of the two boundaries:

- Outside: $r = b$,
  The outer unit normal is $\boldsymbol{n} = \boldsymbol{e}_r$ and in polar coordinates

$$\boldsymbol{n} = n_r\,\boldsymbol{e}_r + n_\theta\,\boldsymbol{e}_\theta + n_z\,\boldsymbol{e}_z, \quad \text{with} \quad n_r = 1, n_\theta = n_z = 0.$$

  The (unwritten) boundary condition is that $\boldsymbol{t} = \boldsymbol{0}$, which in polar coordinates is

$$\boldsymbol{t} = t_r\,\boldsymbol{e}_r + t_\theta\,\boldsymbol{e}_\theta + t_z\,\boldsymbol{e}_z, \quad \text{so that} \quad t_r = t_\theta = t_z = 0.$$

  We now use the continuity of stress condition $t_i = \tau_{ij}\,n_j$, where we can iterate i and j over 'r', 'θ' and 'z' because these are components within an orthonormal coordinate system.

$$\underline{i = r}: \qquad t_r = 0 = \tau_{rr}\,n_r + \tau_{r\theta}\,n_\theta + \tau_{rz}\,n_z = \tau_{rr} \quad \Rightarrow \quad \boxed{\tau_{rr} = 0.}$$

$$\underline{i = \theta}: \qquad t_\theta = 0 = \tau_{\theta r}\,n_r + \tau_{\theta\theta}\,n_\theta + \tau_{\theta z}\,n_z = \tau_{\theta r} \quad \Rightarrow \quad \boxed{\tau_{\theta r} = 0.}$$

$$\underline{i = z}: \qquad t_z = 0 = \tau_{zr}\,n_r + \tau_{z\theta}\,n_\theta + \tau_{zz}\,n_z = \tau_{zr} \quad \Rightarrow \quad \boxed{\tau_{zr} = 0.}$$

- Inside: $r = a$,
  The outer unit normal is $\boldsymbol{n} = -\boldsymbol{e}_r$ and in polar coordinates

$$\boldsymbol{n} = n_r\,\boldsymbol{e}_r + n_\theta\,\boldsymbol{e}_\theta + n_z\,\boldsymbol{e}_z, \quad \text{with} \quad n_r = -1, n_\theta = n_z = 0.$$

  The (written) boundary condition is that $\boldsymbol{t} = -p_0\boldsymbol{n} = p_0\boldsymbol{e}_r$, which in polar coordinates is

$$\boldsymbol{t} = t_r\,\boldsymbol{e}_r + t_\theta\,\boldsymbol{e}_\theta + t_z\,\boldsymbol{e}_z, \quad \text{so that} \quad t_r = p_0, \text{ and } t_\theta = t_z = 0.$$

  Using the same argument as for the outer boundary we obtain the boundary conditions

$$\boxed{\tau_{rr} = -p_0, \quad \tau_{\theta r} = 0, \quad \tau_{zr} = 0.}$$

There is no dependence on either θ or z in the boundary conditions, so it is plausible to try a solution that depends only on r, *i.e.*

$$\boldsymbol{u} = u_r(r)\,\boldsymbol{e}_r. \tag{6.1}$$

Note that if we can't find a solution under this assumption we will have to think again. Note also that because the governing equations are linear, if we do find a solution then it is **the** solutions[1].

We substitute the ansatz, a fancy word for guess, (6.1) into the Navier–Lamé equations in the absence of time-dependent terms and body forces:

$$(\lambda + 2\mu)\,\mathrm{grad}(\mathrm{div}\,\boldsymbol{u}) - \mu\,\mathrm{curl}(\mathrm{curl}\,\boldsymbol{u}) = \boldsymbol{0}.$$

By looking up the formulæ in section 5.5.1, we find that, under this assumption, curl $\boldsymbol{u} = \boldsymbol{0}$. Hence the Navier–Lamé equations become

$$(\lambda + 2\mu)\,\mathrm{grad}(\mathrm{div}\,\boldsymbol{u}) = \boldsymbol{0} \quad \Rightarrow \quad \mathrm{grad}(\mathrm{div}\,[u_r\boldsymbol{e}_r]) = \boldsymbol{0}.$$

[1] This is a slight lie, there could be so-called eigenfunction solutions that satisfy homogenous boundary conditions and introduce non-uniqueness. This is not the case in linear elasticity (in general).

Again, using the expressions in section 5.5.1, we see that only the r-component of the equation is non-trivial and it is

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right] = 0,$$

where the term in square brackets is div$\boldsymbol{u}$. This is a linear, second-order ODE in r and it is straightforward to solve by direct integration:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} = 2A, \quad \text{a constant,}$$

$$\Rightarrow \quad \frac{\partial(ru_r)}{\partial r} = 2Ar \quad \Rightarrow \quad ru_r = Ar^2 + B \quad \Rightarrow \quad \boxed{u_r = Ar + \tfrac{B}{r}.}$$

The final result is the general solution of a second-order ODE, so there are two free constants, as you should expect. We must now apply the boundary conditions at $r = a$ and $r = b$ to determine the unknown constants A and B. If these were displacement conditions, the task would be simple. Unfortunately, our boundary conditions are expressed in terms of the stress. We must, therefore, calculate

$$\tau_{rr} = \lambda \text{div}\boldsymbol{u} + 2\mu\frac{\partial u_r}{\partial r},$$

again see section 5.5.1,

$$\Rightarrow \quad \tau_{rr} = \lambda 2A + 2\mu\left(A - \frac{B}{r^2}\right) \quad \Rightarrow \quad \tau_{rr} = 2A(\lambda+\mu) - 2\mu\frac{B}{r^2}.$$

Applying the boundary conditions gives two simultaneous equations for A and B

$$\underline{r=a} \qquad -p_0 = 2A(\lambda+\mu) - 2B\frac{\mu}{a^2}, \tag{6.2a}$$

$$\underline{r=b} \qquad 0 = 2A(\lambda+\mu) - 2B\frac{\mu}{b^2}. \tag{6.2b}$$

Subtract (6.2a) from (6.2b) to obtain

$$p_0 = -2B\mu\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = 2B\mu\left(\frac{b^2-a^2}{a^2b^2}\right) \quad \Rightarrow \quad \boxed{B = \tfrac{p_0}{2\mu}\tfrac{a^2b^2}{b^2-a^2}.}$$

Then, from equation (6.2b) we have that

$$A = B\frac{\mu}{(\lambda+\mu)b^2} \quad \Rightarrow \quad \boxed{A = \tfrac{p_0}{2(\lambda+\mu)}\tfrac{a^2}{b^2-a^2}.},$$

and so we have an explicit expression for the displacement

$$u_r = \frac{a^2 p_0}{2(b^2-a^2)}\left[\frac{r}{\lambda+\mu} + \frac{b^2}{\mu r}\right].$$

Note that the displacement is directly proportional to the applied pressure and essentially inversely proportional to the Lamé constants.

The "hoop" stress is the stress directed around the circumference of the pipe and given by

$$\tau_{\theta\theta} = \lambda\text{div}\boldsymbol{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \quad \tau_{\theta\theta} = 2\lambda A + 2\mu\left(A + \frac{B}{r^2}\right) = 2A(\lambda+\mu) + \frac{2\mu B}{r^2}.$$

$$\Rightarrow \quad \tau_{\theta\theta} = \frac{a^2 p_0}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right).$$

Thus, we observe that the "hoop" stress decreases with increasing distance though the wall and is greatest the inside wall

$$\tau_{\theta\theta}\big|_{r=a} = \frac{b^2 + a^2}{b^2 - a^2}\, p_0.$$

If the pipe wall is thin $(b - a) \ll 1$ then $\tau_{\theta\theta}\big|_{r=a} \gg p_0$ and the "hoop" stress at the inner wall is much, much greater than the applied pressure.

Note that a force balance about a diameter cut through the pipe implies that

$$2ap_0 = 2\int_a^b \tau_{\theta\theta}\,\mathrm{d}\,r,$$

and you should confirm this result.

6.2 Sphere deformed by self-gravity

If we imagine that a planet is an elastic body, then we can ask the question: by how much does its external radius change due to gravitational attraction of the planet with itself? Lecture 12

We shall assume that the planet is initially a perfect sphere of undeformed radius a and the gravitational body force is directed towards the centre of the planet and proportional to r^2:

$$\boldsymbol{F} = -\frac{\rho g}{a} r \boldsymbol{e}_r,$$

where ρ is the density of the material and g is the acceleration due to gravity, see Figure 6.2.

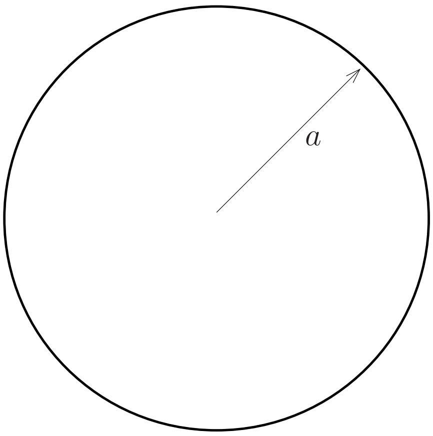


Figure 6.2: An elastic sphere has undeformed radius a and is loaded only by a body force $\boldsymbol{F} = -(\rho g/a) r \boldsymbol{e}_r$.

Here, the natural coordinate system to use is spherical polars (r, θ, ϕ), where

$$\boldsymbol{u} = u_r\,\boldsymbol{e}_r + u_\theta\,\boldsymbol{e}_\theta + u_\phi\,\boldsymbol{e}_\phi.$$

[2]This can be derived from Newton's law of gravitation by considering spherical shells of uniformly distributed mass and integrating up.

There is no external load so $\boldsymbol{t} = \mathbf{0}$ on the outer boundary of the sphere. The boundary conditions and body force are such that we expect the solution to depend only on r, the distance from the centre of the sphere. We therefore choose the ansatz

$$\boldsymbol{u} = u_r \boldsymbol{e}_r,$$

which looks the same as our choice for the previous problem, but, of course, the definition of the radius is different.

Referring to the definitions in section 5.5.2, we find that under this assumption curl $\boldsymbol{u} = \mathbf{0}$ and the Navier–Lamé equations in the absence of time-dependent terms become

$$(\lambda + 2\mu)\,\text{grad}(\text{div}\,\boldsymbol{u}) + \boldsymbol{F} = \mathbf{0}.$$

The only non-trivial component of these equations is in the r direction and is

$$(\lambda + 2\mu)\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r}\right] - \frac{\rho g r}{a} = 0,$$

where the term in square brackets is the divergence of $\boldsymbol{u}$. Once again we have a linear, second-order ODE that can be solved by direct integration. Rearranging, we obtain

$$\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r}\right] = \frac{\rho g}{(\lambda + 2\mu)a} r = kr, \tag{6.3}$$

where we introduce the constant $k = \rho g/((\lambda + 2\mu)a)$ to save writing. Integrating equation (6.3) once with respect to r yields

$$\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} = k\frac{r^2}{2} + \tilde{A} \quad \Rightarrow \quad \frac{\partial(r^2 u_r)}{\partial r} = k\frac{r^4}{2} + \tilde{A}r^2,$$

$$\Rightarrow \quad r^2 u_r = k\frac{r^5}{10} + \tilde{A}\frac{r^3}{3} + \tilde{B} \quad \Rightarrow \quad u_r = Kr^3 + Ar + \frac{B}{r^2},$$

where $K = k/10$ and $A = \tilde{A}/3$ and $B = \tilde{B}$ are unknown constants that must be determined from boundary conditions.

A potential problem is that there are two constants, but only one boundary. However, we also require that the displacement remains *finite* throughout the domain, particularly at $r = 0$. Indeed, we expect that $u_r = 0$ as $r \to 0$, which necessitates that $\tilde{B} = 0$. Hence, we can fix the remaining unknown constant A using the traction-free boundary condition at $r = a$. Again using section 5.5.2, we have

$$\tau_{rr} = \lambda \text{div}\boldsymbol{u} + 2\mu\frac{\partial u_r}{\partial r} = \frac{\lambda}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + 2\mu\frac{\partial u_r}{\partial r},$$

$$\Rightarrow \quad \tau_{rr} = \frac{\lambda}{r^2}\left(5Kr^4 + 3Ar^2\right) + 2\mu\left(3Kr^2 + A\right) \quad \Rightarrow \quad \tau_{rr} = Kr^2\left(5\lambda + 6\mu\right) + A(3\lambda + 2\mu).$$

At $r = a$, we know that $\tau_{rr} = 0$ which means that

$$0 = Ka^2\left(5\lambda + 6\mu\right) + A(3\lambda + 2\mu) \quad \Rightarrow \quad A = -\frac{Ka^2(5\lambda + 6\mu)}{3\lambda + 2\mu},$$

which completely specifies the solution.

For example, we can now calculate the displacement of the surface

$$u_r\big|_{r=a} = Ka^3 - \frac{Ka^3(5\lambda + 6\mu)}{3\lambda + 2\mu} = \frac{Ka^3(3\lambda + 2\mu - 5\lambda - 6\mu)}{3\lambda + 2\mu} = -\frac{Ka^3(2\lambda + 4\mu)}{3\lambda + 2\mu},$$

and after using the definition of K, we obtain

$$u_r\big|_{r=a} = -\frac{\rho g a^2}{5(3\lambda + 2\mu)}, \quad \text{an inward displacement.}$$

Note the the displacement increases with increases in density, gravitational field strength and decreases in λ or μ.

We can also compute the radial "pressure" at the centre of the sphere

$$\tau_{rr}\big|_{r=0} = -Ka^2(5\lambda + 6\mu) = -\frac{\rho g a}{10}\frac{5\lambda + 6\mu}{\lambda + 2\mu},$$

meaning that the pressure remains finite (which is good beacuse it means the planet won't implode!).

6.3 Cylinder supported from below under its own weight

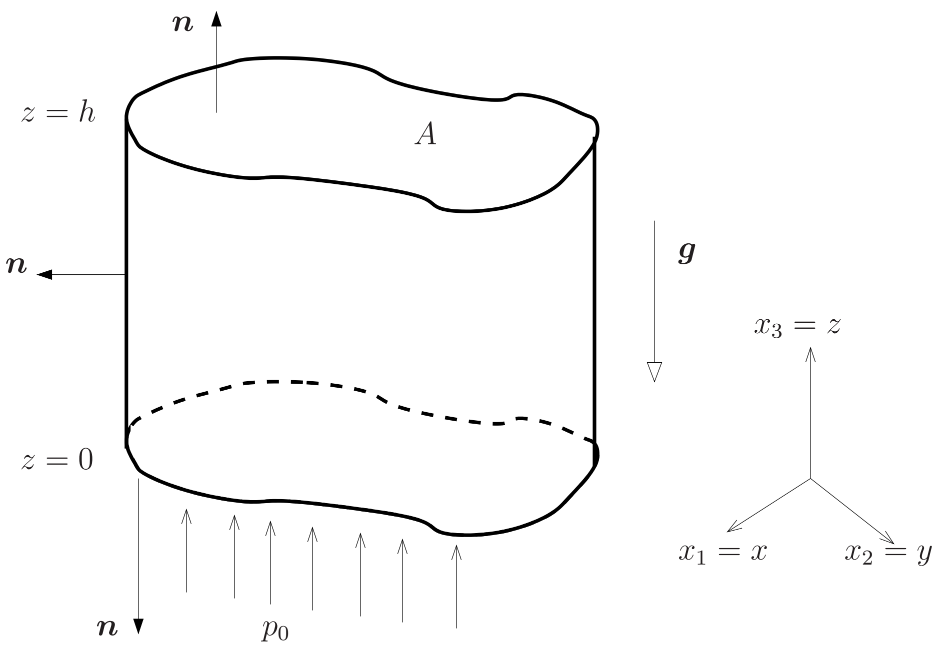


Figure 6.3: A cylinder with arbitrary cross-section A is subject to the gravitational field $\boldsymbol{g}$ aligned with its axis.

We shall consider a cylinder of arbitrary cross-section A that is subject to the body force $\boldsymbol{F} = (0, 0, F_3) = (0, 0, -\rho g)$. The body will fall unless supported by a constant pressure from below p_0, which could be the reaction force from a table or the floor. If the system is to be in overall equilibrium then the forces in the axial (z) direction must balance:

$$A\, p_0 = \rho g V = \rho g h A \quad \Rightarrow \quad p_0 = \rho g h, \quad \text{a hydrostatic balance.}$$

If we consider the boundary conditions on the cylinder then we have

- At the base ($z = 0$):

$$\tau_{33} = -p_0 = -\rho g h.$$

- At the top ($z = h$):

$$\boldsymbol{t} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{n} = (0, 0, 1) \quad \Rightarrow \quad \tau_{13} = \tau_{23} = \tau_{33} = 0.$$

- On the side:

$$\boldsymbol{t} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{n} = (n_1, n_2, 0)$$

$$\Rightarrow \quad \tau_{11}n_1 + \tau_{12}n_2 = 0, \quad \tau_{21}n_1 + \tau_{22}n_2 = 0, \quad \tau_{31}n_1 + \tau_{32}n_2 = 0.$$

There are no displacement boundary conditions.

Physically, we do not expect there to be any transverse stresses τ_{1j}, τ_{2j} within the interior because the cylinder can be chopped into vertical slices without changing the deformation. Thus we shall **try** a solution of the form $\tau_{33} \neq 0$, but $\tau_{ij} = 0$ otherwise. Furthermore, we expect the only variation to be in the axial (z) direction, so we choose the ansatz $\tau_{33}(z)$. The Beltami–Michell equations for constant $\boldsymbol{F}$ reduce to Lecture 13

$$\nabla^2\Theta = \nabla^2(\tau_{11} + \tau_{22} + \tau_{33}) = \nabla^2\tau_{33} = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\tau_{33} = 0 \quad \Rightarrow \quad \tau_{33} = cz + d,$$

where c and d are constants. Using the stress boundary conditions at the top and bottom of the cylinder, we have that

$$\tau_{33}\big|_{z=0} = d = -\rho g h \quad \text{and} \quad \tau_{33}\big|_{z=h} = ch + d = ch - \rho g h = 0 \quad \Rightarrow \quad c = \rho g.$$

Hence, we have an expression for the stress

$$\tau_{33} = \rho g(z - h), \quad \text{and} \quad \tau_{ij} = 0, \quad \text{otherwise}.$$

In order to determine the displacement field we have to work a bit harder. Indeed, it will look somewhat horrific because we shall work in full generality which means that we know there will be six unknown constants in the final solution because we are not suppressing rigid-body motions. There are lots of details, but the process is straightforward. Specifically, we shall use the (inverse) constitutive relationships to find the strain and then solve the resulting PDE system to recover the displacement field. The constitutive law gives

$$Ee_{ij} = (1 + \nu)\tau_{ij} - \nu\delta_{ij}\tau_{kk},$$

Hence, $e_{ij} = 0$ when $i \neq j$, but

$$Ee_{11} = -\nu\tau_{33}, \quad Ee_{22} = -\nu\tau_{33}, \quad Ee_{33} = (1 + \nu)\tau_{33} - \nu\tau_{33} = \tau_{33},$$

and so

$$e_{11} = e_{22} = -\frac{\nu}{E}\tau_{33}, \quad \text{and} \quad e_{33} = \frac{\tau_{33}}{E}.$$

From the definition of the strain tensor, we have the following partial differential equations

$$u_{1,1} = u_{2,2} = -\frac{\nu\rho g}{E}(z - h), \quad \text{and} \quad u_{3,3} = \frac{\rho g}{E}(z - h),$$

from which it follows that

$$u_1 = -\frac{\nu\rho g}{E}(z-h)x + u_0(y,z), \tag{6.4a}$$

$$u_2 = -\frac{\nu\rho g}{E}(z-h)y + v_0(x,z), \tag{6.4b}$$

$$u_3 = \frac{\rho g}{E}\left(\frac{1}{2}z^2 - hz\right) + w_0(x,y). \tag{6.4c}$$

We must now find the unknown functions u_0, v_0 and w_0 from the off-diagonal terms of the strain tensor:

$$e_{12} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u_0(y,z)}{\partial y} + \frac{\partial v_0(x,z)}{\partial x} = 0.$$

The only way that the above expression can be true **for all** values of x, y and z is for both terms to be equal and opposite functions of z, *e.g.*

$$\frac{\partial u_0}{\partial y} = \mathcal{F}(z), \qquad \frac{\partial v_0}{\partial x} = -\mathcal{F}(z). \tag{6.5}$$

$$e_{13} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = 0 \quad \Rightarrow \quad -\frac{\nu\rho g}{E}x + \frac{\partial u_0(y,z)}{\partial z} + \frac{\partial w_0(x,y)}{\partial x} = 0, \tag{6.6}$$

$$e_{23} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = 0 \quad \Rightarrow \quad -\frac{\nu\rho g}{E}y + \frac{\partial v_0(x,z)}{\partial z} + \frac{\partial w_0(x,y)}{\partial y} = 0. \tag{6.7}$$

We can remove the inhomogeneities from the above equations by differentiating equation (6.6) with respect to y and equation (6.7) with respect to x. Firstly, let's differentiate (6.6)

$$\frac{\partial^2 u_0}{\partial y \partial z} + \frac{\partial^2 w_0}{\partial y \partial x} = 0,$$

and using equation (6.5) we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial u_0}{\partial y}\right) + \frac{\partial^2 w_0}{\partial y \partial x} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}z} + \frac{\partial^2 w_0}{\partial y \partial x} = 0 \quad \Rightarrow \quad \frac{\partial^2 w_0}{\partial y \partial x} = -\mathcal{F}'(z). \tag{6.8}$$

Now differentiate (6.7)

$$\frac{\partial^2 v_0}{\partial x \partial z} + \frac{\partial^2 w_0}{\partial y \partial x} = 0,$$

and using equation (6.5) we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial v_0}{\partial x}\right) + \frac{\partial^2 w_0}{\partial y \partial x} = 0 \quad \Rightarrow \quad -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}z} + \frac{\partial^2 w_0}{\partial y \partial x} = 0 \quad \Rightarrow \quad \frac{\partial^2 w_0}{\partial y \partial x} = \mathcal{F}'(z). \tag{6.9}$$

The only way that both equations (6.8) and (6.9) can be true is iff $\mathcal{F}'(z) = 0$. It follows that

$$\frac{\partial^2 w_0}{\partial x \partial y} = 0 \quad \Rightarrow \quad w_0(x,y) = f(x) + g(y)$$

and

$$\mathcal{F}'(z) = 0 \quad \Rightarrow \quad \mathcal{F}(z) = \alpha,$$

where α is a constant. Using these results in equations (6.6) and (6.7) leads to the expressions

$$\frac{\partial u_0(y,z)}{\partial z} = \frac{\nu \rho g}{E} x - f'(x), \tag{6.10a}$$

and

$$\frac{\partial v_0(x,y)}{\partial z} = \frac{\nu \rho g}{E} x - g'(y). \tag{6.10b}$$

Important Note: The prime always means differentiate with respect to the variable and only makes sense for a function of a single variable. What can be confusing is that $f'(x)$ means $\mathrm{d}f/\mathrm{d}x$ and $g'(y)$ means $\mathrm{d}g/\mathrm{d}y$.

In equation (6.10a), the RHS depends only on x, but the LHS depends on y and z. This is only possible for all values of x, y and z if both sides equal the same constant. Thus,

$$\frac{\partial u_0(y,z)}{\partial z} = \beta_1 \quad \text{and} \quad \frac{\nu \rho g x}{E} - f'(x) = \beta_1,$$

and so

$$u_0 = \beta_1 z + h_1(y) \quad \text{and} \quad f(x) = \frac{\nu \rho g}{E}\frac{x^2}{2} - \beta_1 x + \delta_1.$$

Differentiate with respect to y

$$\frac{\partial u_0}{\partial y} = h_1'(y) = \mathcal{F}(z) = \alpha \quad \Rightarrow \quad h_1(y) = \alpha y + \gamma_1,$$

and so

$$u_0 = \alpha y + \beta_1 z + \gamma_1.$$

Similarly, in equation (6.10b), the RHS depends only on y, but the LHS depends on x and z. Once again both sides must equal the same constant.

$$\frac{\partial v_0(y,z)}{\partial z} = \beta_2 \quad \text{and} \quad \frac{\nu \rho g y}{E} - g'(y) = \beta_2,$$

and so

$$v_0 = \beta_2 z + h_2(x) \quad \text{and} \quad g(y) = \frac{\nu \rho g}{E}\frac{y^2}{2} - \beta_2 y + \delta_2.$$

Differentiate with respect to x so

$$\frac{\partial v_0}{\partial x} = h_2'(x) = -\mathcal{F}(z) = -\alpha \quad \Rightarrow \quad h_2(x) = -\alpha x + \gamma_2,$$

and so

$$v_0 = -\alpha x + \beta_2 z + \gamma_2.$$

Believe it or now, we have now finished and have the complete displacement field:

$$u_1 = -\frac{\nu \rho g}{E}(z-h)x + \alpha y + \beta_1 z + \gamma_1, \tag{6.3a}$$

$$u_2 = -\frac{\nu \rho g}{E}(z-h)y - \alpha x + \beta_2 z + \gamma_2, \tag{6.3b}$$

$$u_3 = \frac{\rho g}{E}\left(\frac{1}{2}z^2 - hz\right) + \frac{\nu \rho g}{2E}\left(x^2+y^2\right) - \beta_1 x - \beta_2 y + \gamma_3, \tag{6.3c}$$

where $\gamma_3 = \delta_1 + \delta_2$.

As expected, there are six unknown constraints in the final solution that represent the rigid-body motions.

- The vector $(\gamma_1, \gamma_2, \gamma_3)$ is a translation.
- α is a negative rotation about the z-axis.
- β_1 is a positive rotation about the y-axis.
- β_2 is a negative rotation about the x-axis.

If we fix the origin and suppress rotations then we have the displacement field:

$$u_1 = -\frac{\nu\rho g}{E}(z-h)x, \tag{6.4a}$$

$$u_2 = -\frac{\nu\rho g}{E}(z-h)y, \tag{6.4b}$$

$$u_3 = \frac{\rho g}{E}\left(\frac{1}{2}z^2 - hz\right) + \frac{\nu\rho g}{2E}\left(x^2+y^2\right). \tag{6.4c}$$

We can now use this solution to determine by how much the cylinder has shortened due to the action of gravity. The change in length is given by the difference between the displacements of the top and bottom of the cylinder: $u_3(z=h) - u_3(z=0)$.

$$u_3(z=h) = -\frac{\rho g}{2E}h^2 + \frac{\nu\rho g}{2E}\left(x^2+y^2\right) \quad \text{and} \quad u_3(z=0) = \frac{\nu\rho g}{2E}\left(x^2+y^2\right),$$

so the cylinder is shortened by the amount $\rho g h^2/(2E)$. Note that as you might expect, increasing the density, height and gravitational field strength increases the amount by which the cylinder is shortened, whereas increasing the stiffness, E, decreases the amount of shortening.

Chapter 7

Plane strain problems

7.1 Introduction and Basic Equations

Lecture 14

Definition: A deformation is said to be one of *plane strain* (parallel to the plane $x_3 = 0$) if:

$$u_3 = 0 \quad \text{and} \quad u_1,\ u_2 \text{ are independent of } x_3,\ \textit{i.e. } u_\alpha = u_\alpha(x_\beta), \tag{7.1}$$

where we (should!) recall that the Greek indices α and β only take the values 1 or 2.

There are only two independent variables, $(x_1, x_2) = (x, y)$ and consequently $e_{13} = e_{23} = e_{33} = 0$, which gives the following form for the strain tensor

$$\begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{12} & e_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad d = e_{11} + e_{22},$$

and every component is a function only of x and y.

7.1.1 Stress-strain relations

Recall the constitutive law (4.6) for a linear, isotropic elastic solid

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}.$$

All strain components are functions only of x_1 and x_2 and hence $\tau_{ij}(x_1, x_2)$. It follows immediately that $\tau_{13} = \tau_{23} = 0$, but importantly, in general,

$$\tau_{33} = \lambda(e_{11} + e_{22}) = \lambda d \neq 0.$$

Hence, there may be an out-of-plane stress required to support a state of plane strain.

Note that

$$\Theta = \tau_{kk} = (3\lambda + 2\mu)d = (3\lambda + 2\mu)\tau_{33}/\lambda,$$

but by definition

$$\Theta = \tau_{11} + \tau_{22} + \tau_{33} = \widetilde{\Theta} + \tau_{33},$$

where $\widetilde{\Theta} = \tau_{11} + \tau_{22}$ is the in-plane stress. It follows that

$$(3\lambda + 2\mu)\tau_{33}/\lambda = \widetilde{\Theta} + \tau_{33} \quad \Rightarrow \quad \tau_{33}(3\lambda + 2\mu - \lambda)/\lambda = \widetilde{\Theta} \quad \Rightarrow \quad \tau_{33} = \frac{\lambda}{2(\lambda + \mu)}\widetilde{\Theta} = \nu\widetilde{\Theta},$$

where ν is Poisson's ratio. Thus, although non-zero in general, the out-of-plane stress component τ_{33} is a function of the in-plane stress and because $\Theta = (1+\nu)\widetilde{\Theta}$ is an invariant and $1+\nu$ is a constant, it follows that $\widetilde{\Theta}$ must also be an invariant. We can therefore write the stress tensor corresponding to a state of plane strain in the general form

$$\begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{12} & \tau_{22} & 0 \\ 0 & 0 & \nu(\tau_{11}+\tau_{22}) \end{pmatrix}.$$

The existence of the non-zero τ_{33} term is sometimes known as the Poisson effect and represents the physical result that if you "pull" a thick plate in plane, it will contract unless a vertical pull is applied to keep the plate a fixed thickness.

Examining the inverse constitutive law, we have

$$2\mu e_{ij} = \tau_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}\tau_{kk} = \tau_{ij} = \frac{\lambda(1+\nu)}{3\lambda+2\mu}\,\widetilde{\Theta}\,\delta_{ij} = \tau_{ij} - \nu\,\delta_{ij}\widetilde{\Theta}.$$

In summary a state of plane strain is represented by

$$\tau_{\alpha\beta} = \lambda\delta_{\alpha\beta}e_{\gamma\gamma} + 2\mu e_{\alpha\beta}, \tag{7.2a}$$

$$\tau_{33} = \nu\tau_{\gamma\gamma} = \nu\widetilde{\Theta} = \nu\,\tau_{\gamma\gamma}, \tag{7.2b}$$

$$2\mu e_{\alpha\beta} = \tau_{\alpha\beta} - \nu\,\delta_{\alpha\beta}\,\tau_{\gamma\gamma}. \tag{7.2c}$$

7.1.2 Static Equilibrium Equations

Cauchy's equations are given by

$$\tau_{ij,j} + F_i = 0,$$

but in a state of plane strain $\tau_{ij}(x_\alpha)$ and we know that $\tau_{13} = \tau_{23} = 0$. Hence, the three components of Cauchy's equations become

$$\tau_{11,1} + \tau_{12,2} + F_1 = 0,$$

$$\tau_{21,1} + \tau_{22,2} + F_2 = 0,$$

$$F_3 = 0.$$

It follows that the body force $\boldsymbol{F} = (F_1(x,y), F_2(x,y), 0)$ and it is therefore not possible for there to be a state of plane strain if there are any body forces that act out-of-plane. The non-trivial equilibrium equations are then simply:

$$\tau_{\alpha\beta,\beta} + F_\alpha = 0. \tag{7.3}$$

7.1.3 Strain Compatibility

In general, there are six non-trivial compatibility equations, but when $e_{i3} = 0$ and there is no varation in x_3 direction, there is only one:

$$e_{11,22} + e_{22,11} - 2e_{12,12} = 0. \tag{7.4}$$

We can also formulate this equation in terms of the stresses (*a la* the derivation of the Beltrami–Michell equations): From the constitutive laws

$$2\mu e_{11} = \tau_{11} - \nu\widetilde{\Theta}, \quad 2\mu e_{22} = \tau_{22} - \nu\widetilde{\Theta}, \quad 2\mu e_{12} = \tau_{12},$$

which can be substituted into the compatibility equation (7.4) after multiplication by 2μ to yield

$$(\tau_{11} - \nu\widetilde{\Theta})_{,22} + (\tau_{22} - \nu\widetilde{\Theta})_{,11} - 2\tau_{12,12} = 0,$$

$$\Rightarrow \quad (\widetilde{\Theta} - \tau_{22} - \nu\widetilde{\Theta})_{,22} + (\widetilde{\Theta} - \tau_{11} - \nu\widetilde{\Theta})_{,11} - 2\tau_{12,12} = 0,$$

$$\Rightarrow \quad (1-\nu)\widetilde{\Theta}_{,\alpha\alpha} - [\tau_{11,11} + 2\tau_{12,12} + \tau_{22,22}] = (1-\nu)\widetilde{\Theta}_{,\alpha\alpha} - \tau_{\alpha\beta,\alpha\beta} = 0.$$

Now differentiating the equations of equilibrium (7.3) $-\tau_{\alpha\beta,\alpha\beta} = F_{\alpha,\alpha}$, and so we obtain

$$(1-\nu)\widetilde{\Theta}_{,\alpha\alpha} + F_{\alpha,\alpha} = 0, \tag{7.5}$$

or symbolically

$$(1-\nu)\widetilde{\nabla}^2\widetilde{\Theta} + \text{ div } \mathbf{F} = 0, \tag{7.6}$$

where $\widetilde{\nabla}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

7.1.4 Boundary conditions

Only in-plane traction boundary conditions can be applied to system in which case

$$t_\alpha = \tau_{\alpha\beta} n_\beta.$$

7.2 The Airy stress function

If there are no body force $\boldsymbol{F} = \mathbf{0}$ then the equilibrium equation is Lecture 15

$$\tau_{\alpha\beta,\beta} = 0,$$

and the compatibility equation is

$$\tau_{\alpha\alpha,\beta\beta} = 0,$$

which gives a total of three equations for the three unknown τ_{11}, τ_{22}, τ_{12}. We can reduce these three equations to a single scalar equation as follows.

From the first equilibrium equation

$$\frac{\partial \tau_{11}}{\partial x} = -\frac{\partial \tau_{12}}{\partial y},$$

and so there is a function $\mathcal{F}$ such that

$$\tau_{11} = \frac{\partial \mathcal{F}}{\partial y} \quad \text{and} \quad \tau_{12} = -\frac{\partial \mathcal{F}}{\partial x} \quad \text{(Exercise)}.$$

The choice of where to put the minus sign is simply convention.

From the second equilibrium equation

$$\frac{\partial \tau_{12}}{\partial x} = -\frac{\partial \tau_{22}}{\partial y},$$

and so there is another function $\mathcal{G}$ such that

$$\tau_{12} = -\frac{\partial \mathcal{G}}{\partial y} \quad \text{and} \quad \tau_{22} = \frac{\partial \mathcal{G}}{\partial x} \quad \text{(Exercise)}.$$

Hence,

$$-\tau_{12} = \frac{\partial \mathcal{F}}{\partial x} = \frac{\partial \mathcal{G}}{\partial y},$$

which means that there is yet another function ϕ such that

$$\mathcal{F} = \frac{\partial \phi}{\partial y} \quad \text{and} \quad \mathcal{G} = \frac{\partial \phi}{\partial x},$$

which means that

$$\boxed{\tau_{11} = \frac{\partial^2 \phi}{\partial y^2}, \quad \tau_{22} = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{12} = -\frac{\partial^2 \phi}{\partial x \partial y}.}$$

Choosing the stress component to be consistent with these expression will lead to automatic satisfaction of the equilibrium equations. The function ϕ is called the Airy stress function. Note that ϕ is **not unique** because we can add a linear function of x and y without changing the resulting stress field.

The question is now how do we find ϕ for a given problem. We have yet to use the compatibility condition

$$\tau_{\alpha\alpha,\beta\beta} = \tilde{\boldsymbol{\nabla}}^2 \tilde{\Theta} = 0 \quad \Rightarrow \quad \tilde{\boldsymbol{\nabla}}^2 (\tau_{11} + \tau_{22}) = \tilde{\boldsymbol{\nabla}}^2 \left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} \right) = \tilde{\boldsymbol{\nabla}}^4 \phi = 0.$$

Thus the governing equation for the Airy stress function is the *biharmonic* equation:

$$\tilde{\boldsymbol{\nabla}}^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$

7.2.1 Particular solutions of the biharmonic equation

- Harmonic Functions

  Obviously, all harmonic functions, those for which $\tilde{\boldsymbol{\nabla}}^2 f = 0$, also fulfil the biharmonic equation.

- Polynomials

$$\phi(x, y) = \sum_{i,k} a_{ik} x^i y^k \tag{7.7}$$

  - Any terms with $i + k < 2$ do not give a contribution.
  - Any terms with $i + k < 4$ fulfil $\tilde{\nabla}^4 \phi = 0$ for arbitrary constants a_{ik}. Special cases are:

$\phi(x, y)$	τ_{xx}	τ_{yy}	τ_{xy}	Interpretation:
$a_{02}\ y^2$	$2\ a_{02}$	0	0	constant tension in x-direction
$a_{11}\ xy$	0	0	$-a_{11}$	pure shear
$a_{20}\ x^2$	0	$2\ a_{20}$	0	constant tension in y-direction
$a_{03}\ y^3$	$6\ a_{03} y$	0	0	pure x-bending
$a_{30}\ x^3$	0	$6\ a_{30} x$	0	pure y-bending

  Linear combinations of polynomial terms provide stress fields for combined load cases. For example, a body under pure shear γ with external pressure p has a stress state represented by the Airy stress function

$$\phi = -\gamma xy - \frac{p}{2}\left(x^2 + y^2\right).$$

It follows that

$$\tau_{11} = \frac{\partial^2 \phi}{\partial y^2} = -p, \quad \tau_{22} = \frac{\partial^2 \phi}{\partial x^2} = -p, \quad \tau_{12} = -\frac{\partial^2 \phi}{\partial y \partial x} = \gamma.$$

- Fourier expansions for periodic problems $\phi = \sum_m \phi_m(y) \sin(mx)$ will lead to a series of ODEs for $\phi_m(y)$ that can be solved using standard methods.

7.2.2 Applying the boundary conditions

Lecture 16

We now know that we must solve the biharmonic equation to determine the Airy stress function, but what boundary conditions should we apply? In fact, what boundary conditions are appropriate for the biharmonic equation?

The equation is fourth order, which means that we need two boundary conditions at "each end". At a given boundary, we have two traction conditions $t_\alpha = \tau_{\alpha\beta} n_\beta$, where t_α is given. We should be able to convert these into two boundary conditions on ϕ.

In the most general case, we have an arbitrary boundary (curve) that we can parametrise by the arclength s, always chosen so that $\boldsymbol{t} \times (0, 0, 1) = \boldsymbol{n}$, where $\boldsymbol{t}$ is the tangent vector in the direction of the arclength, see Figure 7.1.

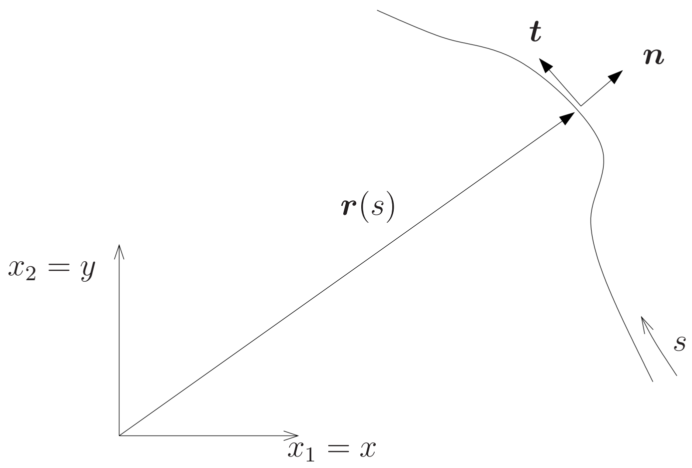


Figure 7.1: An arbitrary curvilinear boundary is described by its position $\boldsymbol{r}(s)$ as a function of the arclength s.

Thus on the boundary ∂D of our domain D we have

$$\boldsymbol{r}(s) = (x_{\partial D}(s), y_{\partial D}(s)),$$

and the tangent vector is formed by differentiating with respect to s

$$\boldsymbol{t}(s) = \left(\frac{\partial x_{\partial D}(s)}{\partial s}, \frac{\partial y_{\partial D}(s)}{\partial s} \right),$$

which is a unit vector. The outer unit normal is then $\boldsymbol{n} = \boldsymbol{t} \times \boldsymbol{e}_z$, so

$$\boldsymbol{n}(s) = \left(\frac{\partial y_{\partial D}(s)}{\partial s}, -\frac{\partial x_{\partial D}(s)}{\partial s} \right).$$

Example

To help fix these ideas consider a circle of radius R centred on the origin. The position vector is $\boldsymbol{r} = (R\cos\theta, R\sin\theta)$, and for a given angle θ the arclength $s = R\theta$, so $\theta = s/R$. Thus the position is

$$\boldsymbol{r}(s) = (R\cos(s/R), R\sin(s/R)) \quad \Rightarrow \quad \boldsymbol{t} = (-\sin(s/R), \cos(s/R), \quad \text{a unit vector as claimed;}$$

and

$$\boldsymbol{n} = (\cos(s/R), \sin(s/R) = \boldsymbol{r}/R, \quad \text{again as expected.}$$

With this background in mind, let us return to the traction boundary conditions and use the Airy stress function and expression for the normal

$$t_1(s) = \tau_{11}n_1 + \tau_{12}n_2 = \frac{\partial^2\phi}{\partial y^2}\frac{\partial y}{\partial s} - \frac{\partial^2\phi}{\partial x\partial y}\left(-\frac{\partial x}{\partial s}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\phi}{\partial y}\right),$$

because by the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial^2\phi}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2\phi}{\partial x\partial y}\frac{\partial x}{\partial s}.$$

Similarly we can deduce that

$$t_2(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\phi}{\partial x}\right).$$

Thus we have two equations for the boundary conditions

$$t_1(s) = t_x(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\phi}{\partial y}\right) \tag{7.8a}$$

and

$$t_2(s) = t_y(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\phi}{\partial x}\right). \tag{7.8b}$$

Hence, if $t_\alpha(s)$ is given, the boundary conditions for ϕ can be derived by the following procedure:

1. Integrate (7.8a) and (7.8b) along the boundary (w.r.t. s). This provides $(\partial\phi/\partial x, \partial\phi/\partial y)^T = \widetilde{\nabla}\phi$ on the boundary.

2. Rewrite $\tilde{\nabla}\phi = \partial\phi/\partial s\ \mathbf{e}_t + \partial\phi/\partial n\ \mathbf{e}_n$ where $\mathbf{e}_t$ and $\mathbf{e}_n$ are the unit tangent and (outer) normal vectors on the boundary. This provides $\partial\phi/\partial s$ and $\partial\phi/\partial n$ along the boundary.

3. Finally, integrate $\partial\phi/\partial s$ along the boundary (w.r.t. s). This provides ϕ along the boundary.

After this procedure ϕ and $\partial\phi/\partial n$ are known along the entire boundary and can be used as the boundary conditions for the fourth-order biharmonic equation.

Notes

- Any constants of integration arising during the procedure can be set to zero, corresponding to setting $\phi = 0$ and $\partial\phi/\partial n = 0$ at particular points on the boundary.
- For a traction free boundary, $t_\alpha(s) = 0$, we can use the boundary conditions:

$$\phi = 0 \quad \text{and} \quad \partial\phi/\partial n = 0 \quad \text{on } \partial D. \tag{7.9}$$

7.2.3 Recovering the displacements from ϕ

Let $\boldsymbol{u} = (u_1, u_2, 0) = (u, v, 0)$, then we start from the constitutive equation in inverse form

$$2\mu e_{\alpha\beta} = \tau_{\alpha\beta} - \nu\delta_{\alpha\beta}\,\tau_{\gamma\gamma}.$$

When $\alpha = \beta = 1$, we have

$$2\mu e_{11} = 2\mu\frac{\partial u}{\partial x} = \tau_{11} - \nu(\tau_{11} + \tau_{22}) + \tau_{22} - \tau_{22} = (1-\nu)(\tau_{11} + \tau_{22}) - \tau_{22}.$$

Then using the expressions for the stresses in terms of the Airy stress function, we obtain

$$2\mu\frac{\partial u}{\partial x} = (1-\nu)\tilde{\nabla}^2\phi - \frac{\partial^2\phi}{\partial x^2}, \tag{7.10}$$

Similarly when $\alpha = \beta = 2$, we obtain

$$2\mu\frac{\partial v}{\partial y} = (1-\nu)\tilde{\nabla}^2\phi - \frac{\partial^2\phi}{\partial y^2} \tag{7.11}$$

and when $\alpha = 1$, $\beta = 2$, the resulting equation is

$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{\partial^2\phi}{\partial x\partial y}. \tag{7.12}$$

These can be integrated directly or via complex variables (not covered in this course).

7.2.4 Equations in polar coordinates

- The biharmonic equation in polar coordinates:

$$\tilde{\nabla}^4\phi(r,\varphi) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}\right] \tag{7.13}$$

$$\tilde{\nabla}^4\phi(r,\varphi) = \phi_{,rrrr} + \frac{2}{r}\phi_{,rrr} - \frac{1}{r^2}(\phi_{,rr} - 2\phi_{,rr\varphi\varphi}) + \frac{1}{r^3}(\phi_{,r} - 2\phi_{,r\varphi\varphi}) + \frac{1}{r^4}(4\phi_{,\varphi\varphi} + 2\phi_{,\varphi\varphi\varphi\varphi}) \tag{7.14}$$

- For axisymmetric solutions:

$$\tilde{\nabla}^4\phi(r) = \frac{1}{r}\left[r\left(\frac{1}{r}[r\phi_{,r}]_{,r}\right)_{,r}\right]_{,r} \tag{7.15}$$

$$\tilde{\nabla}^4\phi(r) = \phi_{,rrrr} + \frac{2}{r}\phi_{,rrr} - \frac{1}{r^2}\phi_{,rr} + \frac{1}{r^3}\phi_{,r} \tag{7.16}$$

- Stresses:

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}, \tag{7.17}$$

$$\tau_{\varphi\varphi} = \frac{\partial^2\phi}{\partial r^2} \tag{7.18}$$

$$\tau_{r\varphi} = \frac{1}{r^2}\frac{\partial\phi}{\partial\varphi} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\varphi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\varphi}\right). \tag{7.19}$$

Solutions in polar coordinates

- The general axisymmetric solution:

$$\phi(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \tag{7.20}$$

- The general separated non-axisymmetric solution:

  For $n = 1$:

$$\begin{aligned}\phi(r,\varphi) = &\left(Ar + \frac{B}{r} + Cr^3 + Dr\log r\right) \quad \cos(\varphi)\\ &+\left(ar + \frac{b}{r} + cr^3 + dr\log r\right) \quad \sin(\varphi)\end{aligned} \tag{7.21}$$

  For $n \geq 2$:

$$\begin{aligned}\phi(r,\varphi) = \sum_{n=2}^{\infty} &\left(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}\right) \quad \cos(n\varphi)\\ &+\left(a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}\right) \quad \sin(n\varphi)\end{aligned} \tag{7.22}$$

7.3 Example: Periodic loading on an elastic half-plane

A semi-infinite, plane elastic body occupies the half-space $y \leq 0$ and is loaded by the external traction $\boldsymbol{t} = -\sin(2\pi x)\boldsymbol{e_y}$ at its upper surface, $y = 0$, see Figure 1. Lecture 17

At the boundary $y = 0$, we have $t_\alpha = \tau_{\alpha\beta}n_\beta$ and the outer unit normal is given by $\boldsymbol{n} = (0, 1)$. (Note that, as usual in a linear elasticity framework, the outer unit normal is that of the undeformed body, for which the upper surface is the plane $y = 0$.) Thus,

$$\boldsymbol{t} = \begin{pmatrix} 0 \\ -\sin(2\pi x)\end{pmatrix} = \begin{pmatrix} \tau_{11}n_1 + \tau_{12}n_2 \\ \tau_{21}n_1 + \tau_{22}n_2\end{pmatrix} = \begin{pmatrix}\tau_{12} \\ \tau_{22}\end{pmatrix}.$$

The Airy stress function Φ is defined such that $\partial^2\Phi/\partial x^2 = \tau_{22}$ and so

$$\frac{\partial^2\Phi}{\partial x^2} = -\sin(2\pi x), \quad \text{at } y = 0.$$

$$\Rightarrow \quad \Phi = \frac{1}{4\pi^2}\sin(2\pi x) + g(y);$$

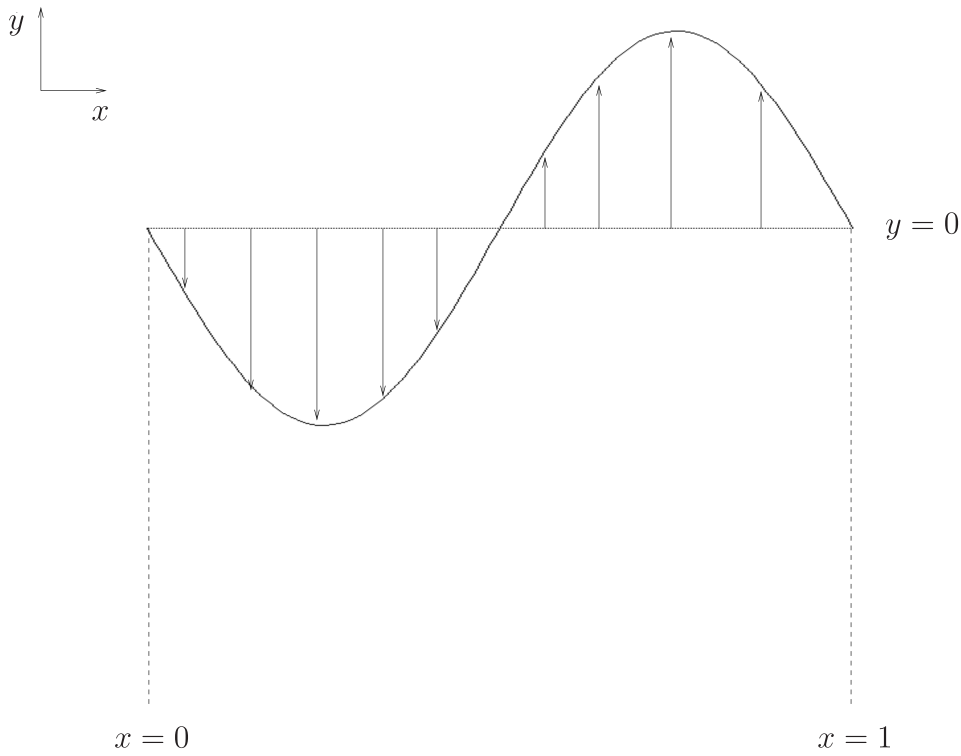


Figure 7.2: A sinusoidal loading of the form $-\sin(2\pi x)\boldsymbol{e_y}$ is applied to the elastic half-space $y \leq 0$.

and, for simplicity, we neglect the arbitrary function $g(y)$[1].

Motivated by this boundary condition we **try**

$$\Phi = \frac{1}{4\pi^2}\sin(2\pi x)f(y),$$

where the function $f(y)$ represents the variations in the vertical direction.

The Airy stress function satisfies the biharmonic equation $\tilde{\nabla}^4\Phi = 0$:

$$\begin{aligned} \frac{\partial^4\Phi}{\partial x^4} \quad &+ \quad 2\frac{\partial^4\Phi}{\partial^2 x\partial^2 y} \quad + \quad \frac{\partial^4\Phi}{\partial y^4} \quad = \quad 0, \\ \Rightarrow \quad 4\pi^2\sin(2\pi x)f(y) \quad &+ \quad -2\sin(2\pi x)f''(y) \quad + \quad \tfrac{1}{4\pi^2}\sin(2\pi x)f^{iv}(y) \quad = \quad 0. \end{aligned} \tag{7.23}$$

Equation (7.23) must be valid for all x and so, after multiplication by $4\pi^2$,

$$f^{iv} - 8\pi^2 f'' + 16\pi^4 f = 0. \tag{7.24}$$

The equation (7.24) is a homogeneous, linear, fourth-order, ODE with constant coefficients and so we pose a solution of the form $f = \mathrm{e}^{ny}$. On substitution of our solution into equation (7.24) we obtain

$$n^4\mathrm{e}^{ny} - 8\pi^2 n^2\mathrm{e}^{ny} + 16\pi^4\mathrm{e}^{ny} = 0, \tag{7.25}$$

which must be true for all y; so

$$n^4 - 8\pi^2 n^2 + 16\pi^4 = 0, \quad \text{(the characteristic equation)}.$$

[1]If we cannot find a solution after this simplification we must revisit this assumption

$$\Rightarrow (n^2 - 4\pi^2)^2 = 0 \qquad \Rightarrow n = \pm 2\pi, \pm 2\pi,$$

and we have two pairs of repeated roots.

The general solution of (7.24) (with four linearly independent functions) is, therefore,

$$f(y) = A\mathrm{e}^{2\pi y} + B\mathrm{e}^{-2\pi y} + Cy\mathrm{e}^{2\pi y} + Dy\mathrm{e}^{-2\pi y}.$$

We expect all quantities to decay as $y \to -\infty$, which implies that B and D are both zero. Thus, our Airy stress function has the form

$$\Phi = \frac{1}{4\pi^2}(A + Cy)\mathrm{e}^{2\pi y}\sin(2\pi x). \tag{7.26}$$

Here, we keep the general form[2] (7.26) and calculate the displacements from the equations

$$2\mu\frac{\partial u}{\partial x} = (1-\nu)\tilde{\nabla}^2\Phi - \frac{\partial^2\Phi}{\partial x^2}, \tag{7.27a}$$

$$2\mu\frac{\partial v}{\partial y} = (1-\nu)\tilde{\nabla}^2\Phi - \frac{\partial^2\Phi}{\partial y^2}, \tag{7.27b}$$

$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{\partial^2\Phi}{\partial x\partial y}. \tag{7.27c}$$

From equation (7.26), we have that

$$\Phi_{xx} = -(A + Cy)\mathrm{e}^{2\pi y}\sin(2\pi x), \quad \text{and} \quad \Phi_{yy} = (A + Cy)\mathrm{e}^{2\pi y}\sin(2\pi x) + \frac{C}{\pi}\mathrm{e}^{2\pi y}\sin(2\pi x).$$

$$\text{and} \quad \Phi_{xy} = (A + Cy)\mathrm{e}^{2\pi y}\cos(2\pi x) + \frac{C}{2\pi}\mathrm{e}^{2\pi y}\cos(2\pi x).$$

and

$$\tilde{\nabla}^2\Phi = \Phi_{xx} + \Phi_{yy} = \frac{C}{\pi}\mathrm{e}^{2\pi y}\sin(2\pi x).$$

Thus, equation (7.27a) becomes

$$2\mu\frac{\partial u}{\partial x} = \left\{(1-\nu)\frac{C}{\pi} + A + Cy\right\}\mathrm{e}^{2\pi y}\sin(2\pi x),$$

$$\Rightarrow u = -\frac{1}{4\mu\pi}\left\{(1-\nu)\frac{C}{\pi} + A + Cy\right\}\mathrm{e}^{2\pi y}\cos(2\pi x) + G_1(y).$$

Equation (7.27b) becomes

$$2\mu\frac{\partial v}{\partial y} = -\left\{\nu\frac{C}{\pi} + A + Cy\right\}\mathrm{e}^{2\pi y}\sin(2\pi x),$$

$$\Rightarrow v = -\frac{1}{4\pi\mu}\left\{\left(\nu - \frac{1}{2}\right)\frac{C}{\pi} + A + Cy\right\}\mathrm{e}^{2\pi y}\sin(2\pi x) + G_2(x).$$

[2]In the lectures I assumed that $C = 0$, which makes the algebra easy but does not satisfy the boundary condition $\tau_{12} = 0$ at the upper surface. I then argued that the resultant horizontal force at $y = 0$ was the same as the real problem that we wanted to solve, and so the solution is correct in a St. Venant sense.

Finally, equation (7.27c) becomes

$$\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=-\left(\frac{C}{2\pi}+A+Cy\right)\mathrm{e}^{2\pi y}\cos(2\pi x).$$

$$\Rightarrow\quad \mu\left(-\frac{1}{2\mu}\left\{(1-\nu)\frac{C}{\pi}+A+Cy+\frac{C}{2\pi}\right\}\mathrm{e}^{2\pi y}\cos(2\pi x)+\frac{\partial G_1}{\partial y}(y)\right.$$
$$\left.-\frac{1}{2\mu}\left\{\left(\nu-\frac{1}{2}\right)\frac{C}{\pi}+A+Cy\right\}\mathrm{e}^{2\pi y}\cos(2\pi x)+\frac{\partial G_2}{\partial x}(x)\right)$$
$$=-(\frac{C}{2\pi}+A+Cy)\mathrm{e}^{2\pi y}\cos(2\pi x).$$

$$\Rightarrow\quad \frac{\partial G_1}{\partial y}(y)+\frac{\partial G_2}{\partial x}(x)=0. \tag{7.28}$$

The equation (7.28) is the same as derived in lectures (it must be otherwise we would not get the appropriate arbitrary rigid-body motion) and because the first term is a function only of y and the second only of x it follows that

$$\frac{\partial G_1}{\partial y}=-\frac{\partial G_2}{\partial x}=K,\quad \text{a constant.}$$

Thus,

$$G_1(y)=Ky+\alpha,\quad \text{and}\quad G_2(x)=-Kx+\beta.$$

Hence, we have that the general solution for the displacement is

$$u = -\frac{1}{4\mu\pi}\left\{(1-\nu)\frac{C}{\pi}+A+Cy\right\}\mathrm{e}^{2\pi y}\cos(2\pi x)+Ky+\alpha, \tag{7.29a}$$

$$v = -\frac{1}{4\pi\mu}\left\{\left(\nu-\frac{1}{2}\right)\frac{C}{\pi}+A+Cy\right\}\mathrm{e}^{2\pi y}\sin(2\pi x)-Ky+\beta, \tag{7.29b}$$

where K is a rigid body rotation within the plane and (α,β) is an in-plane translation.

Applying the boundary conditions at the upper surface

The normal traction component $\tau_{22}=-\sin(2\pi x)$ at $y=0$. Now,

$$\tau_{22}=\Phi_{xx}=-(A+Cy)\mathrm{e}^{2\pi y}\sin(2\pi x),$$

and so

$$\tau_{22}|_{y=0}=-A\sin(2\pi x)=-\sin(2\pi x)\quad\Rightarrow\quad A=1.$$

The tangential traction component is

$$\tau_{12}=-\Phi_{xy}=-\left(A+Cy+\frac{C}{2\pi}\right)\mathrm{e}^{2\pi y}\cos(2\pi x),$$

and so

$$\tau_{12}|_{y=0}=-\left(1+\frac{C}{2\pi}\right)\cos(2\pi x),$$

It follows that we can choose the tangential traction to be exactly zero if $C = -2\pi$. We are restricted in our choice of tangential tractions, however; we can only choose tangential tractions of the functional form $K\cos(2\pi x)$.

The resultant tangential force at the surface is, therefore, always

$$F_x = \int_0^1 \tau_{12}\,\mathrm{d}x = \int_0^1 K\cos(2\pi x)\,\mathrm{d}x = \left[\frac{K}{2\pi}\sin(2\pi x)\right]_0^1 = 0.$$

Thus, there is never any resultant tangential force at the surface, which is consistent with the periodic nature of the loading.

Error analysis

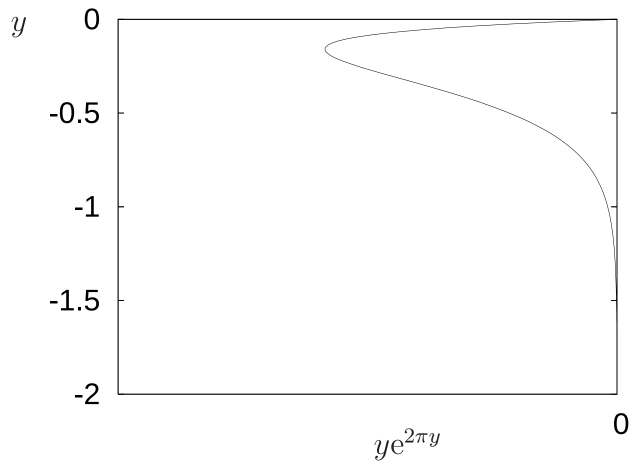
The error in the displacements between the cases when $C = 0$ and $C = -2\pi$ is

$$\text{error in } u = \frac{1}{2\mu\pi}\left\{(1-\nu) + \pi y\right\} \mathrm{e}^{2\pi y}\cos(2\pi x),$$

and

$$\text{error in } v = \frac{1}{2\mu\pi}\left\{(\nu - 1/2) + \pi y\right\} \mathrm{e}^{2\pi y}\sin(2\pi x).$$

The dominant contribution to the error is of the form $\frac{1}{2\mu} y\mathrm{e}^{2\pi y}$, which has a maximum at $y = -1/(2\pi)$, suggesting that dominant errors are confined to a thin layer near the surface of depth approximately equal to the wavelength of the periodic load.



7.4 Example: Circular hole in plate under tension

An infinite elastic plate $-\infty < x < \infty$, $-\infty < y < \infty$ contains a hole of radius a whose centre is chosen to be the origin of our coordinate system. A uniform tension of magnitude T is applied in the x_1 direction. If there were no hole present then the stress τ_{11} would be T everywhere, but we wish to determine the effect of the hole's presence. In particular, we wish to find the "hoop" stress at the hole. The problem is interesting because we will need to work in a mixture of Cartesian and polar coordinates to solve it. Lecture 18

The stress boundary conditions in the far field are most easily expressed in Cartesian components.

$$\tau_{xx} \to T, \tau_{xy} \to 0, \quad \text{and} \quad \tau_{yy} \to 0, \quad \text{as} \quad r \to \infty,$$

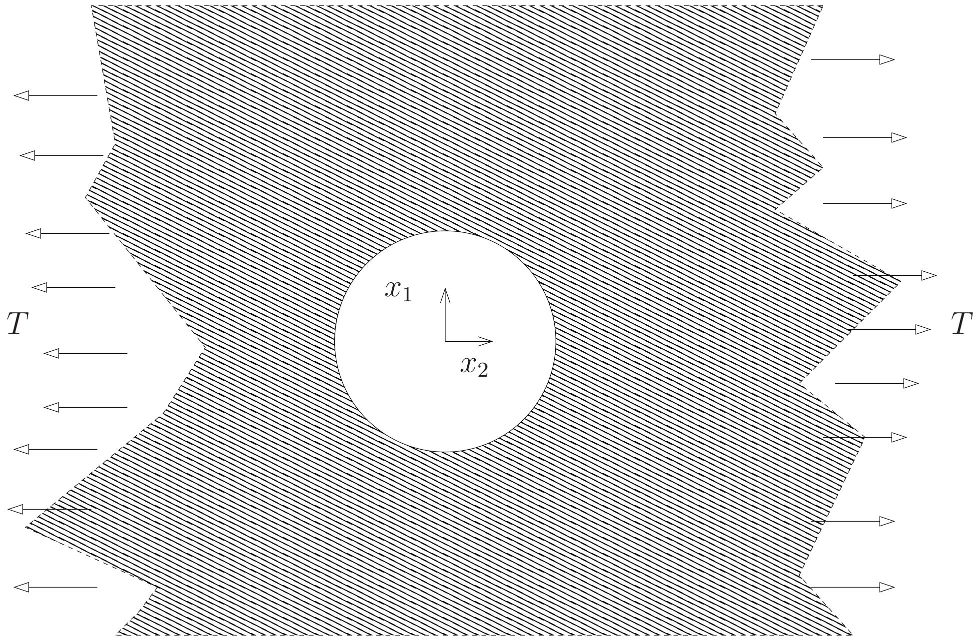


Figure 7.3: An infinite plate of isotropic, linearly elastic material contains a hole of radius a and is subject to a tension T in the x_1 direction.

and that there is no traction at the hole in polars

$$\tau_{rr} = \tau_{r\theta} = 0, \quad \text{at} \quad r = a.$$

We seek an Airy stress function ϕ that describes the stress field and we will need to work in both coordinate systems

$$\begin{array}{ll}
\text{Cartesian} & \text{Polar} \\
\tau_{xx} = \dfrac{\partial^2 \phi}{\partial y^2}, & \tau_{rr} = \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r}, \\
\tau_{yy} = \dfrac{\partial^2 \phi}{\partial x^2}, & \tau_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2}, \\
\tau_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y}, & \tau_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}\right).
\end{array}$$

First consider the far-field boundary condition

$$\tau_{xx} = \frac{\partial^2 \phi}{\partial y^2} \to T \quad \text{as} \quad y, r \to \infty,$$

which means that

$$\phi \to \frac{1}{2}Ty^2 = \frac{1}{2}Tr^2 \sin^2\theta = \frac{1}{4}Tr^2\left(1 - \cos 2\theta\right), \quad \text{as} \quad r \to \infty.$$

We have used the double-angle formula in the last step for reasons that will become clear soon.

Notes

- $\widehat{\phi} = \frac{1}{4}Tr^2(1 - \cos 2\theta)$ satisfies the biharmonic equation.
- The stress components τ_{xy} and τ_{yy} corresponding to $\widehat{\phi}$ as an Airy stress function are both zero, because $\widehat{\phi}$ does not depend on x.

Thus, $\widehat{\phi}$ satisfies all the stress boundary conditions in the far field. Unfortunately, it does not satisfy the boundary conditions at the hole, so we need to work a bit harder!

We can now examine the general separated solution of the biharmonic equation in polar coordinates, see section 7.2.4. In order to match to the far-field solution we need only the "$n = 0$" and $\cos 2\theta$ "$n = 2$" terms. We used the double-angle formula precisely so that we could (easily) make this identification. Thus, we try as our candidate solution

$$\phi = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r + \left(A_2 r^2 + B_2 r^{-2} + C_2 r^4 + D_2 r^0\right)\cos(2\theta).$$

Matching with the condition as $r \to \infty$ immediately tells us that $B_0 = \frac{1}{4}T$ and $A_2 = -\frac{1}{4}T$. The constant term A_0 will have no effect on the stress so we can set it to zero, and we have

$$\phi = \frac{1}{4}Tr^2(1 - \cos 2\theta) + C_0 \ln r + D_0 r^2 \ln r + \left(B_2 r^{-2} + C_2 r^4 + D_2 r^0\right)\cos(2\theta).$$

The terms $r^2 \ln r$ and $r^4 \cos(2\theta)$ will lead to infinite stresses at infinity because $\tau_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$ which tends to ∞ as $r \to \infty$ if either of these two terms are included. This is not physical and therefore we set $D_0 = C_2 = 0$. Hence,

$$\phi = \frac{1}{4}Tr^2(1 - \cos 2\theta) + A \ln r + \frac{B\cos(2\theta)}{r^2} + C\cos(2\theta).$$

We have three unknowns A, B and C so we require three boundary conditions.

The stress boundary conditions at the hole ($r = a$) are given by $t_\alpha = 0 = \tau_{\alpha\beta} n_\beta$ and $\boldsymbol{n} = -\boldsymbol{e}_r$, so

$$\tau_{rr}\big|_{r=a} = 0, \quad \text{and} \quad \tau_{r\theta}\big|_{r=a} = 0,$$

seemingly only two boundary conditions, so is there a problem? The answer is no because

$$\tau_{rr}\big|_{r=a} = \left.\left(\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right)\right|_{r=a},$$

will be of the form $\alpha + \beta\cos(2\theta)$ for two constants α and β. From the boundary condition

$$\alpha + \beta\cos(2\theta) = 0, \quad \text{for all values of } \theta,$$

which means that both $\alpha = 0$ and $\beta = 0$ — two conditions. Similarly,

$$\tau_{r\theta}\big|_{r=a} = \left.\left(-\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial \theta}\right)\right)\right|_{r=a},$$

will be of the form $\gamma\sin(2\theta)$ for constant γ. From the boundary condition

$$\gamma\sin(2\theta) = 0, \quad \text{for all values of } \theta,$$

which means that $\gamma = 0$. The (messy) conditions $\alpha = 0$, $\beta = 0$ and $\gamma = 0$ are three simultaneous equations for the three unknowns A, B, C. Deriving and solving them (exercise), we find that

$$A = -\frac{1}{2}Ta^2, \quad B = -\frac{1}{4}Ta^4, \quad C = \frac{1}{2}Ta^2,$$

and then

$$\tau_{rr} = \frac{T}{2}\left[1 - \left(\frac{a}{r}\right)^2 + \left(1 - 4\left(\frac{a}{r}\right)^2 + 3\left(\frac{a}{r}\right)^4\right)\cos(2\theta)\right],$$
$$\tau_{\theta\theta} = \frac{T}{2}\left[1 + \left(\frac{a}{r}\right)^2 - \left(1 + 3\left(\frac{a}{r}\right)^4\right)\cos(2\theta)\right],$$
$$\tau_{r\theta} = -\frac{T}{2}\left[1 + 2\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4\right]\sin(2\theta).$$

The stress component $\tau_{\theta\theta}$ is the "hoop" stress, the circumferential stress around the hole and when $r = a$,

$$\tau_{\theta\theta}\big|_{r=a} = \frac{T}{2}\left[2 - 4\cos(2\theta)\right].$$

This is maximised when $\theta = \pi/2,\ 3\pi/2$, in which case $\tau_{\theta\theta} = 3T$. In other words the stress at the top and bottom of the hole is three times larger than the stress applied "at infinity" — this factor of three is called the stress intensity factor. You can conduct an experiment to test part of this theory. If it is correct then when you pull a sheet of paper with a hole in the middle then it should rip at the top and bottom of the hole, relative to the direction of pull. Try it and find out!

7.5 Example: Cantilever beam under constant load

Consider a linearly elastic beam of thickness $2H$ attached at one end to a wall and loaded above by a constant load q, see Figure 7.4. This is a simple model for a shelf attached to a wall. Lecture 19

We seek an Airy stress function ϕ such that $\tilde{\nabla}^4\phi = 0$ and the stress boundary conditions on each face are satisfied.

- Face I

$$\tau_{yy} = -q, \quad \tau_{xy} = 0.$$

- Face II

$$\tau_{xx} = 0, \quad \tau_{xy} = 0.$$

- Face III

$$\tau_{yy} = 0, \quad \tau_{xy} = 0.$$

Unlike previous examples where we solved the biharmonic equation for the Airy stress function directly. In this example, we shall assemble a plausible stress function constructively by thinking about the possible variations in τ_{ij}.

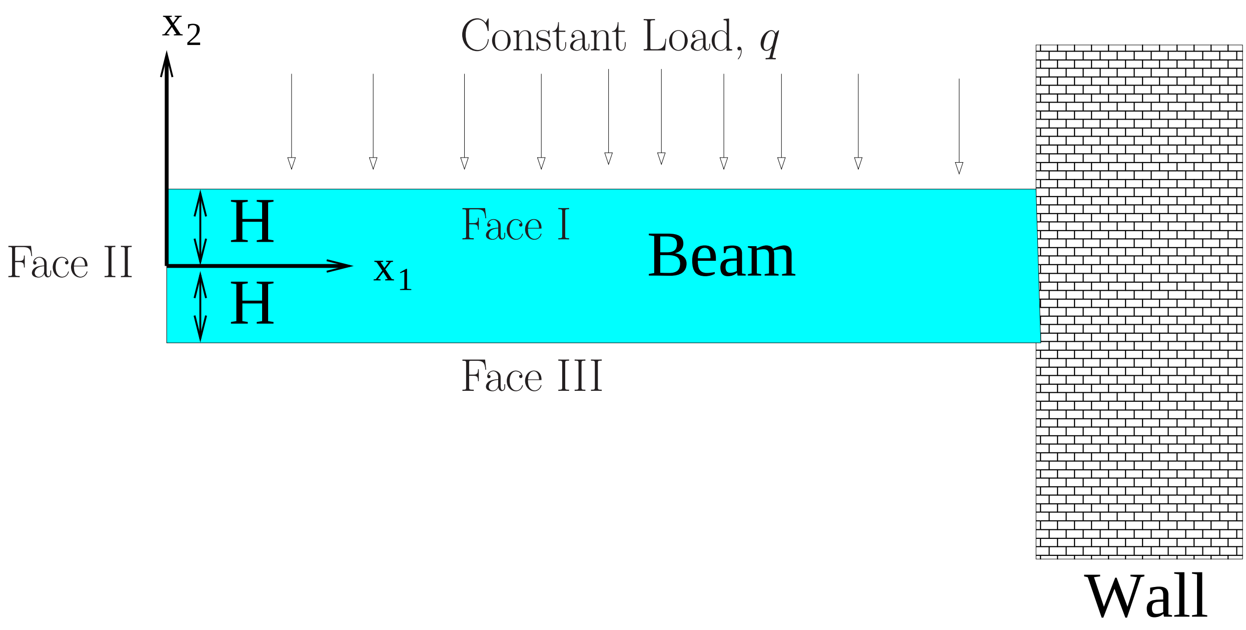


Figure 7.4: A beam of width $2H$ is attached to a wall at its right-hand end and loaded from above by a constant load q.

Shear stresses

Consider an overall vertical force balance on a section of the beam starting from the right-hand (free) end of length x. The total downward force is qx from the upper surface. The balancing upward force is given only by τ_{xy} at the surface x (because $\tau_{xy} = 0$ on Face II). Thus we require

$$qx = \int_{-H}^{H} \tau_{xy} \, \mathrm{d}y,$$

which implies that τ_{xy} must vary linearly with x; and so, because $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$, it follows that $\phi \sim x^2$ to give the required linear variation. Moreover, $\tau_{xy} = 0$ at $y = \pm H$, but is not identically zero throughout the beam. Thus, we must have at least quadratic variation in y, which leads to three possible terms in the Airy stress function

$$\phi \sim x^2(A\, y + B\, y^2 + C\, y^3) \quad \Rightarrow \quad \tau_{xy} \sim x(A + 2B\, y + 3C\, y^2).$$

Bending

Once again consider a section of the beam of length x, but now examine the horizontal force balance. The only horizontal force is provided by τ_{xx} on the surface x, which means that for equilibrium

$$\int_{-H}^{H} \tau_{xx} \, \mathrm{d}y = 0.$$

Thus, τ_{xx} must be an odd function of y, so we cannot have the $x^2 y^2$ term proposed above. Now we consider a moment balance. The anticlockwise moment on the top surface must be balanced by a clockwise moment at the right-hand side

$$\int qx\,\mathrm{d}x = \int_{-H}^{H} \tau_{xx} y\,\mathrm{d}y,$$

$$\Rightarrow \quad q\frac{x^2}{2} = \int_{-H}^{H} \tau_{xx} y\,\mathrm{d}y,$$

which means that $\tau_{xx} = \frac{\partial^2}{\partial y^2} \sim x^2$. The simplest variation is linear in y, which suggests that we require a term of the form $\phi \sim x^2 y^3$. This is consistent with terms deduced in the previous section for τ_{xy}. We can also add a constant bending moment $\tau_{xx} \sim y$ to fixed boundary conditions.

Vertical stress

From the boundary conditions we know that τ_{yy} must vary from $-q$ at $y = H$ to 0 at $y = -H$. The simplest way to satisfy these two criteria is to have a linear variation in τ_{yy} between the top and bottom of the beam. This requires terms in the Airy stress function of the form

$$\phi \sim Dx^2 y + Ex^2 \quad \Rightarrow \quad \tau_{yy} = \frac{\partial^2 \phi}{\partial x^2} \sim 2Dy + 2E.$$

Putting all these requirements together we obtain the candidate Airy stress function

$$\phi = ax^2 + bx^2 y + cx^2 y^3 + dy^3.$$

We must now check that it actually satisfies the biharmonic equation

$$\widetilde{\nabla}^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 24cy \neq 0.$$

We can remove the offending term by adding a new term of the form ey^5, in which case

$$\widetilde{\nabla}^4 \phi = 24cy + 120ey,$$

and in order to satisfy the biharmonic equation

$$24c + 120e = 0 \quad \Rightarrow \quad e = -\frac{1}{5}c.$$

Note that this is not the only way of solving the problem, but throughout the entire process we have been trying to find the simplest possible Airy stress function that has the required ingredients.

Hence, we take

$$\phi = ax^2 + bx^2 y + c\left(x^2 y^3 - \frac{1}{5}y^5\right) + dy^3,$$

as a possible Airy stress function that satisfies the biharmonic equation and has the "right" physics. It remains to be seen whether we can satisfy the boundary conditions.

Lecture 20

From our Airy stress function, we have

$$\tau_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 6cx^2y - 4cy^3 + 6dy,$$

$$\tau_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 2a + 2by + 2cy^3,$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -(2bx + 6cxy^2).$$

Let us now apply the boundary conditions

- Face I

$$\tau_{yy} = -q \quad \text{at} \quad y = H,$$
$$-q = 2\left(a + bH + cH^3\right)). \tag{7.30a}$$
$$\tau_{xy} = 0 \quad \text{at} \quad y = H,$$
$$0 = 2bx + 6cxH^2. \tag{7.30b}$$

- Face III

$$\tau_{yy} = 0 \quad \text{at} \quad y = -H,$$
$$0 = 2\left(a - bH - cH^3\right). \tag{7.30c}$$
$$\tau_{xy} = 0 \quad \text{at} \quad y = -H,$$
$$0 = 2bx + 6cxH^2,$$

  which is simply equation (7.30b) again because we have already built this symmetry into our Airy stress function.

Summing the equations (7.30a) and (7.30c) yields

$$-q = 4a \quad \Rightarrow \quad \boxed{a = -\tfrac{1}{4}q}.$$

From equation (7.30b) we have

$$b = -3cH^2, \tag{7.30d}$$

and using (7.30d) in equation (7.30c), after dividing by two, yields

$$0 = -\frac{1}{4}q + 3cH^3 - cH^3 \quad \Rightarrow \quad \frac{1}{4}q = 2H^3c \quad \Rightarrow \quad \boxed{c = \tfrac{1}{8}\tfrac{q}{H^3};}$$

and then using (7.30d) gives

$$\boxed{b = -\tfrac{3}{8}\tfrac{q}{H}.}$$

We must still find the unknown d, but we have not yet used the boundary conditions on Face II. The condition that $\tau_{xy} = 0$ at $x = 0$ is satisfied by construction, so we have one boundary condition

$$\tau_{xx} = 0, \quad \text{at} \quad x = 0;$$

and one unknown coefficient. There is a problem, however, because

$$\tau_{xx} = -4cy^3 + 6dy = 6dy - \frac{1}{2}\frac{q}{H^3}y^3,$$

which cannot be *identically* zero (zero for all values of y) no matter the choice of d because the functional forms y and y^3 are not the same.

Instead, we appeal to St. Venant's principle

> *St. Venant's principle*
>
> In elastostatics, if the boundary tractions $\mathbf{t}$ on a part ∂D_1 of the boundary ∂D are replaced by a statically equivalent traction distribution $\hat{\mathbf{t}}$, the effects on the stress distribution in the body are negligible at points whose distance from ∂D_1 is large compared to the maximum distance between the points of ∂D_1.

'Statically equivalent' means that the resultant forces and moments due to the two tractions $\mathbf{t}$ and $\hat{\mathbf{t}}$ are identical. Hence, the traction boundary conditions are not fulfilled pointwise but in an average sense.

For example a point load of magnitude $\boldsymbol{F}$, a traction $\boldsymbol{t}$ such that $\int \boldsymbol{t}\,\mathrm{d}y = \boldsymbol{F}$ or a suspended weight of mass such that $m\boldsymbol{g} = \boldsymbol{F}$ are all statically equivalent. The idea is that far away from the boundary all these loads should cause the same deformation, see the discussion at the end of section 7.3.

In the present case, we must calculate the resultant force in the x-direction

$$F_x = \int_{-H}^{H} \tau_{xx}\,\mathrm{d}y = 0$$

because τ_{xx} is odd in y, which is already statically equivalent to the desired boundary condition. The resultant moment is

$$M_x = \int_{-H}^{H} \tau_{xx} y\,\mathrm{d}y = \int_{-H}^{H} (6dy - \frac{1}{2}\frac{q}{H^3}y^3)y\,\mathrm{d}y = \int_{-H}^{H} 6dy^2 - \frac{1}{2}\frac{q}{H^3}y^4\,\mathrm{d}y$$

$$= \left[2dy^3 - \frac{1}{10}\frac{q}{H^3}y^5\right]_{-H}^{H} = 4dH^3 - \frac{1}{5}\frac{q}{H^3}H^5 = 4dH^3 - \frac{1}{5}qH^2.$$

Thus, in order to have a statically equivalent loading, we require $M_x = 0$, and hence

$$\boxed{d = \frac{1}{20}\frac{q}{H}.}$$

Putting all the pieces together we have the strain energy function

$$\phi = -\frac{1}{4}qx^2 - \frac{3}{8}\frac{q}{H}x^2y + \frac{1}{8}\frac{q}{H^3}\left(x^2y^3 - \frac{1}{5}y^5\right) + \frac{1}{20}\frac{q}{H}y^3.$$

This provides a good approximation for the stress field within our beam and can be used to determine the resultant forces at the wall.