Please submit solutions to **all** questions by **1:00pm** on Wednesday 29th November 2017. Hand in to reception in the Alan Turing Building.

Linear (uncoupled) thermoelasticity

In the standard formulation of elasticity the stress (τ_{ij}) is a function of the strain (e_{ij}) only. Hence, for an isotropic, homogeneous linearly-elastic material, in the absence of any pre-stress, the constitutive law is

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}, \tag{1}$$

where δ_{ij} is the Kronecker delta and λ and μ are the Lamé constants.

In thermoelasticity, the effects of temperature on the stress are also taken into account. In a homogeneous, isotropic material increasing the temperature causes the material to expand uniformly. In the linear formulation the constitutive law becomes:

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\delta_{ij}, \tag{2}$$

where α is the rate of thermal expansion, $T(\boldsymbol{x})$ is the temperature field and T_0 is a constant reference temperature.

1. **Understanding the constitutive law** (4 marks)
   (a) Use equation (2) to show that the body is unstressed when it is unstrained and the temperature is T_0.
   (b) Find the strain tensor that leads to a state of zero stress at constant temperature T. What type of physical deformation is represented by this strain tensor?

2. **Governing equations (displacement form)** (4 marks)

   Use the definition of strain and the equations of motion from the lecture notes with the thermoelastic constitutive law (2) to deduce the thermoelastic equivalent of the steady Navier–Lamé equations in the absence of any body forces:

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} - (3\lambda + 2\mu)\alpha \boldsymbol{\nabla} T = \boldsymbol{0}, \tag{3}$$

   where $\boldsymbol{u}$ is the displacement vector.

3. **Heating a pipe** (12 marks)
   A circular pipe of a linearly-elastic material with constitutive law given by equation (2) has undeformed inner radius a and outer radius b. The pipe is full of stationary hot water maintained at a temperature T_1 and the external air temperature is T_0. The inner and outer walls of the pipe are stress free.
   (a) Find the temperature field within the pipe walls, assuming that the temperature field obeys a steady diffusion equation, $\nabla^2 T = 0$, and is a function only of r, the distance from the axis of the pipe.
   (b) Hence, use equation (3) to find the displacement of the pipe. State clearly any simplifying assumptions made. (**Warning:** The answer gets quite messy because there are a large number of constants. You may wish to check your answer using a computer algebra package.)
   (c) Produce a graph of the temperature distribution and displacement field throughout the pipe for $a = 1$, $b = 2$, $T_0 = 0$, $T_1 = 1$, $\lambda = \mu = \alpha = 1$. Explain the result physically.

You may use the results that in cylindrical polar coordinates (r, θ, z),

$$\boldsymbol{u} = u_r \hat{\boldsymbol{r}} + u_\theta \hat{\boldsymbol{\theta}} + u_z \hat{\boldsymbol{z}},$$

where $\hat{\boldsymbol{r}}$ is the unit vector in the r direction, etc, and

$$\text{grad}\, f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \text{div}\, \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\text{curl}\, \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

$$\tau_{rr} = \lambda \text{div}\, \boldsymbol{u} + 2\mu \frac{\partial u_r}{\partial r} - (3\lambda + 2\mu)\alpha(T - T_0).$$