Two hours

THE UNIVERSITY OF MANCHESTER

ELASTICITY

18 January 2018

14:00 – 16:00

Answer **ALL FIVE** questions.

University approved calculators may be used.

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1. A linearly elastic body is subject to the displacement field

$$u_1 = 5 + \alpha x_3 + x_2, \quad u_2 = \beta x_1 - 3x_1 - 4x_3, \quad u_3 = \gamma(x_2 + x_1),$$

where α, β and γ are constants.

(i) Find the strain and rotation tensors associated with the displacement field. Is the deformation homogeneous?

[6 marks]

(ii) Find values of the constants α, β and γ such that the displacement consists of rigid-body motions only.

[4 marks]

(iii) If $\beta = 2$ and $\alpha = -\gamma$, find the principal strains in terms of γ and the corresponding principal axes of strain.

[6 marks]

2.

(i) Explain why a tensor a_{ij} with the non-zero entries

$$a_{11} = x_1, \quad a_{12} = 7x_2, \quad a_{21} = 7x_3, \quad a_{22} = x_2,$$

cannot be a valid strain tensor.

[1 mark]

(ii) Write down the entries of a strain tensor that corresponds to all material lines in $x_1 - x_2$ planes being stretched by a factor of $(1 + \epsilon)$, but no strain in the x_3 direction.

[3 marks]

(iii) Calculate the entries of the corresponding stress tensor for a homogeneous, isotropic linearly elastic material with constitutive law

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

where λ and μ are the Lamé coefficients.

[2 marks]

You may make use the strain compatibility equations

$$e_{ij,kl} + e_{kl,ij} - e_{kj,il} - e_{il,kj} = 0,$$

without proof.

3. An infinite elastic plate $-\infty < x < \infty$, $-\infty < y < \infty$ contains a hole of radius a whose centre is chosen to be the origin of the Cartesian coordinate system. Far from the hole, a uniform tension of magnitude T is applied in both the x and y directions.

(i) Write down the stress boundary conditions on the hole boundary and as $r \to \infty$, where r is the distance from the origin of the coordinate system.

[3 marks]

(ii) Write down the equation satisfied by the Airy stress function.

[1 mark]

(iii) Find the Airy stress function and corresponding stress field that satisfies the boundary conditions found in (i). What is the maximum value of the "hoop stress", $\tau_{\theta\theta}$ in the plate?

[14 marks]

You may use the fact that in plane polar coordinates the stress components may be recovered from an Airy stress function $\Phi(r,\theta)$ via

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}, \quad \tau_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta};$$

and also that the general separated solution of the biharmonic equation in plane polar coordinates (r,θ) can be written in the form

$$A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r + \left(Ar + \frac{B}{r} + Cr^3 + Dr\log r\right)\cos(\theta) + \left(ar + \frac{b}{r} + cr^3 + dr\log r\right)\sin(\theta)$$

$$+\sum_{n=2}^{\infty}\left(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}\right)\cos(n\theta) + \left(a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}\right)\sin(n\theta)$$

4. A bathysphere is a diving vessel that consists of a spherical shell of a linearly elastic material with inner radius a and outer radius b. The internal boundary is stress free and the external pressure may by assumed to be a constant P. All body forces can be neglected.

(i) Assuming that the resulting displacements can be written in the form

$$\boldsymbol{u} = u_r(r)\,\hat{\mathbf{r}},$$

where (r, θ, ϕ) are spherical polar coordinates and $\hat{\mathbf{r}}$ is a unit vector directed away from the centre of the sphere, solve the Navier–Lamé equations

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} + \boldsymbol{F} = \mathbf{0},$$

to find the general solution for the displacement field throughout the sphere.

[5 marks]

(ii) Write down the stress boundary conditions at the outer and inner boundaries $r = a$ and $r = b$.

[2 marks]

(iii) Find the radial displacement field $u_r(r)$ and the stress field throughout the bathysphere's wall.

[10 marks]

(v) The bathysphere will fail if the magnitude of any of the internal stress components τ_{ij}, exceeds the value σ. Find the maximum value of P that can be sustained by the bathysphere.

[3 marks]

You may use the results that in spherical polar coordinates (r, θ, ϕ)

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}},$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\,u_r) + \frac{\partial}{\partial \theta}(r\sin\theta\,u_\theta) + \frac{\partial}{\partial \phi}(ru_\phi)\right\},$$

$$\operatorname{curl}\mathbf{u} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(u_\phi\sin\theta) - \frac{\partial u_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r}(ru_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.$$

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \qquad \tau_{\theta\theta} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right),$$

$$\tau_{\phi\phi} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial \phi} + u_r + u_\theta\cot\theta\right), \qquad \frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta},$$

$$\frac{\tau_{r\phi}}{\mu} = \frac{\tau_{\phi r}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}, \qquad \frac{\tau_{\theta\phi}}{\mu} = \frac{\tau_{\phi\theta}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi\cot\theta}{r}.$$

5. A circular cylinder of radius R is made of a linearly elastic material with Lamé constants λ and μ and constitutive law

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}.$$

The cylinder is twisted about its axis such that the steady applied displacement field is of the form

$$u_1 = -\Omega x_2 x_3, \quad u_2 = \Omega x_1 x_3, \quad u_3 = \Omega\, \psi(x_1, x_2),$$

where Ω is a constant, $\psi(x_1, x_2)$ is an unknown function and x_i are the components of position in a Cartesian coordinate system with the x_3 coordinate directed along the cylinder's axis.

(i) Show that only non-zero terms in the stress tensor are $\tau_{13} = \tau_{31}$ and $\tau_{23} = \tau_{32}$ and that writing them in the form

$$\tau_{13} = \mu\Omega \frac{\partial \Upsilon}{\partial x_2}, \quad \tau_{23} = -\mu\Omega \frac{\partial \Upsilon}{\partial x_1},$$

for some stress function $\Upsilon(x_1, x_2)$, satisfies the equations of motion

$$\tau_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

[6 marks]

(ii) Find equations that relate $\psi(x_1, x_2)$ and $\Upsilon(x_1, x_2)$ and hence show that $\nabla^2 \Upsilon = -2$.

[3 marks]

(iii) If the outer curved surface of the cylinder is stress free, by using polar coordinates or otherwise, explain why $\Upsilon = 0$ is an appropriate boundary condition on the curved surface.

[4 marks]

(iv) Find Υ assuming that it is a function only of $r = \sqrt{x_1^2 + x_2^2}$ and that $\Upsilon = 0$ on the curved surface.

[5 marks]

(v) Hence find the torsional rigidity of the cylinder, which is given by the integral of $2\mu\Upsilon$ over the cylinder's cross-section.

[2 marks]

You may use the results that in cylindrical polar coordinates (r, θ, z)

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div} \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl} \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

$$\tau_{rr} = \lambda \operatorname{div} \mathbf{u} + 2\mu \frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda \operatorname{div} \mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda \operatorname{div} \mathbf{u} + 2\mu \frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

END OF EXAMINATION PAPER