Two hours

THE UNIVERSITY OF MANCHESTER

ELASTICITY

17 January 2017

09:45 – 11:45

Answer **ALL FIVE** questions.

University approved calculators may be used.

---

1. An elastic sphere of undeformed radius 1 is subjected to the displacement field

$$u_1 = -\epsilon x_2, \quad u_2 = 2\epsilon\, x_1, \quad u_3 = 2\epsilon\, x_3,$$

where ϵ is a small positive constant, and u_i is the displacement component in direction of the Cartesian coordinate x_i.

(i) Find the strain and rotation tensors associated with the displacement field. Is the deformation homogeneous?

[6 marks]

(ii) Find the principal strains and principal axes of strain and hence describe the deformation.

[6 marks]

(iii) By using the results from parts (i) and (ii), or otherwise, sketch the deformed configuration in the plane $x_3 = 0$.

[4 marks]

2. Consider the two-dimensional Cartesian strain tensor with entries

$$e_{11} = \sin x_2, \quad e_{12} = f(x_1, x_2), \quad e_{22} = x_1.$$

(i) Explain why a valid two-dimensional displacement field cannot be recovered from the strain tensor when $f(x_1, x_2) \equiv 0$.

[4 marks]

(ii) Find the most general form of the function $f(x_1, x_2)$ such that a valid displacement field can be recovered from the strain tensor; and calculate a corresponding displacement field in the case when $f(x_1, x_2) = \frac{1}{2} x_1 \cos x_2$.

[8 marks]

You may make use the strain compatibility equations

$$e_{ij,kl} + e_{kl,ij} - e_{kj,il} - e_{il,kj} = 0,$$

without proof.

3.

(i) Describe what is meant by a constitutive law for an elastic material and explain the assumptions that lead to the linear constitutive law:

$$\tau_{ij} = \tau_{ij}^0 + E_{ijkl} e_{kl}, \tag{1}$$

carefully defining each term.

[5 marks]

(ii) Use the constitutive law (1) as well as the definition of the strain tensor to write the equations of motion,

$$\tau_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2},$$

in terms of u_i, E_{ijkl} and τ_{ij}^0. Do not make any assumptions about the form of E_{ijkl}.

[3 marks]

(iii) Confirm that the governing equation found in (ii) reduces to the Navier–Lamé equation

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u},$$

when

$$E_{ijkl} = \lambda\, \delta_{ij} \delta_{kl} + 2\mu\, \delta_{ik}\, \delta_{jl},$$

where λ and μ are the usual Lamé constants, $F_i = 0$ and $\tau_{ij}^0 = 0$.

[4 marks]

4. A finite cylinder of undeformed length l is made of a linearly elastic material of density ρ; Lamé constants λ and μ; and undeformed radius 1. The cylinder rotates at a constant angular velocity $\omega > 0$ about its axis which means that in a frame rotating with the cylinder, the governing equations are the Navier–Lamé equations

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} + \boldsymbol{F} = \mathbf{0},$$

where $\boldsymbol{F} = \rho\omega^2 r\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}$ is a radial unit vector directed away from the axis of the cylinder. The curved surface of the cylinder is subjected to a constant external pressure P and the other surfaces (the ends) are fixed in the axial direction in the rotating frame.

(i) Assuming that the resulting displacements can be written in the form

$$\boldsymbol{u} = u_r(r)\,\hat{\mathbf{r}} + u_z(z)\,\hat{\mathbf{z}},$$

where (r, θ, z) are cylindrical polar coordinates and $\hat{\mathbf{z}}$ is a unit vector directed along the axis of the cylinder, solve the Navier–Lamé equations to find the general solution for the displacement field throughout the cylinder.

[6 marks]

(ii) Write down the stress boundary condition at the outer curved boundary $r = 1$.

[2 marks]

(iii) Use the fixed-end boundary conditions to find $u_z(z)$ throughout the cylinder.

[2 marks]

(iv) Find the radial displacement field $u_r(r)$ throughout the cylinder.

[4 marks]

(v) Hence, find the external pressure P required to prevent the outer surface of the cylinder from expanding. Describe the variation in P as the angular velocity increases. Finally, find a simplified expression for the displacement field u_r for the value of P found above and sketch u_r as a function of r.

[6 marks]

You may use the results that in cylindrical polar coordinates (r, θ, z)

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div} \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl} \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

$$\tau_{rr} = \lambda \operatorname{div} \mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda \operatorname{div} \mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda \operatorname{div} \mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

5. An infinite linearly elastic solid under plane strain contains a semi-infinite crack in the $x_1 - x_2$ plane along the negative x_1-axis. Introducing the usual plane polar coordinate system in this plane, the two sides of the crack are defined to be at $\theta = \pm\pi$.

(i) Write down the equation that is satisfied by the Airy Stress function Φ.

[2 marks]

(ii) Assuming that the boundaries of the crack at $\theta = \pm\pi$ are traction free explain why the boundary conditions are given by $\tau_{\theta\theta} = \tau_{r\theta} = 0$.

[3 marks]

(iii) By posing an Airy stress function of the form $\Phi = r^{n+1} g(\theta)$, show that

$$g(\theta) = A_1 \cos(n+1)\theta + A_{-1} \cos(n-1)\theta + B_1 \sin(n+1)\theta + B_{-1} \sin(n-1)\theta,$$

where A_1, A_{-1}, B_1 and B_{-1} are unknown constants.

[6 marks]

(iv) By applying the boundary conditions on the crack find all possible values of n that lead to non-trivial stress fields.

[6 marks]

(v) Hence, find the value of n that leads to the slowest decay of stresses as $r \to \infty$. For this value of n find an expression for $\tau_{\theta\theta}$ that involves only two unknown constants, A_1 and B_1.

[3 marks]

You may use the results that in plane polar coordinates:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad 2e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

$$\nabla^2 f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2};$$

and for an Airy stress function, $\Phi(r,\theta)$

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}, \quad \tau_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta}.$$

You may also wish to use the identity

$$\sin A \pm B = \sin A \cos B \pm \cos A \sin B.$$

END OF EXAMINATION PAPER