Most students attempted every question on the exam and the majority did well. There were relatively few algebra and calculus problems, although index notation did cause problems for a few (particularly in question 3). Overall the weakest areas were sketching and physical interpretation of results.

1 This question tested knowledge of strain and rotation tensors, as well as principal strains and axes of strain. It also tested sketching and the ability to give physical interpretation to deformations. The parts of the question that students found most difficult were sketching, eigenvalue/eigenvector calculations and physical interpretation. Many students did not answer the question about axisymmetry.

(i) The most common mistake was to work only with the corners and to assume that all sides remain straight, whereas the upper and lower surface of the deformed cylinder are curved, which can be seen from u_3. The deformation **is** axisymmetric. Surprisingly few student explicitly wrote down the expression for the deformed position, as requested.

(ii) The vast majority of students obtained full marks here. There were a few algebraic slips and missing factors of a half. Remember that the strain tensor is symmetric (by construction) and the rotation tensor is antisymmetric. A minority of students assumed that the tensors were two-dimensional instead of three-dimensional.

(iii) The vast majority of people correctly stated that the principal strains are the eigenvalues and correctly calculated the eigenvalues. Many did more work than was required because the eigenvalues of a diagonal matrix are simply the diagonal entries. Some forgot that this is a three-dimensional problem, so there should be three eigenvalues! The description of the deformation was missed out by quite a few people, which (of course) lost marks.

2 This question tested knowledge of Airy stress functions, how they are related to the stress and how the stress is related to the traction. It again tested sketching. Almost everybody correctly determined that Φ was a valid Airy stress function, but finding the stress field, boundary tractions and sketching the loads proved problematic. Once you know the Airy stress function, the stress field follows from the relationships:

$$\tau_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \tau_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}.$$

Once you know the stresses you can determine the tractions from the relationship $t_i = \tau_{ij} n_j$. Nearly everybody correctly computed the normal vectors, but not all used the fact that the boundaries are defined by specific values of the coordinates, $x = 0$ or 1 and $y = 0$ or 1.

3 This question tested the ability to solve equations in index notation, as well as concepts of plane strain and plane stress.

(i) The majority of students correctly rearranged the equations to find

$$F_i = -\tau_{ij,j} = -\tau_{i1,1} - \tau_{i2,2},$$

because there is no dependence on x_3. Hence, $F_1 = -a$, $F_2 = -c \cos x_2$ and $F_3 = 0$. Common mistakes were forgetting the minus signs and no including F_3. The body is three-dimensional despite the fact that it is in a state of plane stress.

(ii) This question could be answered by following the standard steps to derive the constitutive law in inverse form, as in the lecture notes. The most common error was not to remember that one needs to find the relationship between the traces of the stress and strain tensors

$$\tau_{kk} = (3\lambda + 2\mu) e_{kk},$$

before you can find e_{ij} as an explicit function of τ_{ij}.

(iii) The body is not in a state of plane strain because e_{33} is not zero. I also accepted general statements about the Poisson effect if correctly argued.

4 This question tested whether students understood how to derive a new stress function, as well as how to construct boundary conditions for stress functions.

(i) This was well answered by the majority and follows directly from the definition.

(ii) Again this was well-answered and follows directly from the given formula.

(iii) This part required students to use the results from (ii) about the form of the stress tensor and the fact that no body forces were acting in the equilibrium equations. The first two equilibrium equations ($i = 1, 2$) are automatically satisfied, and the third ($i = 3$) can be satisfied by a stress function of the given form. I think everybody has the correct idea, but some were let down by algebraic slips. The equation (1) follows directly by differentiating the explicit expressions for the stress components found in (ii).

(iv) This was probably the least well answered question on the entire exam. Nobody gave the perfect solution. Many tried to reproduce the arguments from the lecture notes about the Airy stress function (which is the right idea), but this is a different stress function. The outer unit normal to the cylinder side only has components in the x and y directions so for it to be traction-free we have

$$\tau_{xx} n_x + \tau_{xy} n_y = 0, \quad \tau_{yx} n_x + \tau_{yy} n_y = 0, \quad \tau_{zx} n_x + \tau_{zy} n_y = 0.$$

The first two equations are automatically satisfied by the last becomes:

$$\frac{\partial \phi}{\partial y} n_x - \frac{\partial \phi}{\partial x} n_y = 0.$$

For an arclength coordinate the outer unit normal is given by

$$n_x = \frac{\partial y}{\partial s}, n_y = -\frac{\partial x}{\partial s},$$

and so

$$\frac{\partial \phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s} = \frac{\mathrm{d}\phi}{\mathrm{d}s} = 0,$$

after using the chain rule.

(v) This was generally well answered. The argument is simply that if $\mathrm{d}\phi/\mathrm{d}s = 0$ on the boundary then that means that ϕ is constant on the boundary and the chosen form has $\phi = 0$ on the boundary, which is certainly a constant. The value of C follows from direct substitution into the equation

$$\nabla^2 \phi = -2\mu\theta.$$

(vi) This last part proved to be surprisingly tricky. The only non-zero stresses are $\tau_{xz} = \partial\phi/\partial y = 2yC/b^2$ and $\tau_{yz} = -\partial\phi/\partial x = -2Cx/a^2$, using the given form for ϕ in part (v). The square of the stress magnitude is then

$$|\tau|^2 = 4C^2 \left(\frac{y^2}{b^4} + \frac{x^2}{a^4}\right),$$

which takes its maximum value on the boundary when $y = \pm b$ and $x = 0$.

5 Almost every student scored at least 50% of the marks on this question. The problem is essentially the same one solved in spherical and cylindrical coordinates and is similar to examples seen in lectures and on example sheets. The most common mistakes were neglecting the wrong term in the solution of the resulting ODE. The elastic domain does not include the origin, but the displacement must remain bounded as $r \to \infty$, so the terms that are proportional to positive powers of r must be removed and those proportional to negative powers of r retained.

Comparing the displacements at $r = a$ leads one to conclude that the cylinder has a greater deformed radius — its displacement is twice as large at the sphere. For the final part the most common mistake was to forget that the deformed radius is the sum of the initial radius and the displacement.