Two hours

THE UNIVERSITY OF MANCHESTER

ELASTICITY

15 January 2013

09:45 – 11:45

Answer **ALL SIX** questions.

Electronic calculators may be used, provided that they cannot store text.

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1. A two-dimensional elastic body has an undeformed configuration given by a diamond defined by the vertices $(x_1, x_2) = (-1, 0)$, $(1, 0)$, $(0, 1)$ and $(0, -1)$. The body is subjected to the displacement field

$$u_1 = \epsilon\,(x_2^2 + x_1), \qquad u_2 = \epsilon\, x_1^2,$$

where ϵ is a small positive constant.

(i) Calculate the displacement of the vertices and plot them on a graph. Do the straight lines between the vertices remain straight after the deformation.

(ii) Determine the strain tensor e_{ij} and the rotation tensor ω_{ij} corresponding to this displacement field.

(iii) Find the maximum extensional strain within the body and the position(s) at which it acts.

[12 marks]

2. Explain why the function

$$\Psi(x, y) = x^3 y^2$$

is not a valid Airy stress function. Find a function $g(y)$ so that

$$\Phi(x, y) = \Psi(x, y) + x g(y),$$

is a valid Airy stress function. Is your choice of $g(y)$ unique?

[6 marks]

3. An elastic body is subjected to a small displacement $\boldsymbol{u}$.

(i) A parallelepiped within the undeformed body, formed by the three infinitesimal vectors $(\mathrm{d}x_1, 0, 0)$, $(0, \mathrm{d}x_2, 0)$ and $(0, 0, \mathrm{d}x_3)$, has volume $\mathrm{d}v$. Show that the corresponding deformed volume is given by

$$\mathrm{d}V = (1 + u_{i,i})\mathrm{d}v,$$

if terms quadratic in the displacement gradient $u_{i,j}$ are neglected.

You may use the fact that the volume of a parallelepiped formed by three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is given by the absolute value of the scalar triple product $|\boldsymbol{a}\cdot(\boldsymbol{b} \times \boldsymbol{c})|$.

(ii) Let the density of the material in the undeformed configuration be given by ρ_0 and the density in the deformed configuration be ρ, so that the total mass of the deformed region is $\rho\mathrm{d}V$. If the total mass of the region is conserved during the deformation show that

$$\frac{\rho - \rho_0}{\rho} = -u_{i,i}.$$

(iii) By using the constitutive law

$$\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij},$$

and the result from (ii) establish the relationship

$$P(\rho) = K(\rho - \rho_0)/\rho_0,$$

where $P = -\tau_{kk}/3$ and $K = \lambda + 2\mu/3$.

[8 marks]

4. A sphere made of a linearly elastic material has undeformed radius a and Lamé constants λ and μ. The sphere's surface is subjected to a prescribed displacement

$$\boldsymbol{u}(r=a) = b\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is a unit vector directed away from the centre of the sphere.

(i) By assuming a displacement field of the form $\boldsymbol{u} = f(r)\,\hat{\mathbf{r}}$ find the displacement field throughout the sphere.

You may use the fact that the displacement field satisfies the Navier–Lamé equations in the absence of any body force

$$(\lambda + 2\mu)\,\text{grad}\,(\text{div}\,\boldsymbol{u}) - \mu\,\text{curl}\,(\text{curl}\,\boldsymbol{u}) = \mathbf{0}.$$

(ii) Calculate the induced stress τ_{rr} at the centre of the sphere.

(iii) What is the surface pressure required to achieve the deformation? Give a physical explanation for the sign of the pressure by considering the case when $b < 0$.

[12 marks]

You may use the results that in spherical polar coordinates (r, θ, ϕ)

$$\text{grad}\, f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}},$$

$$\text{div}\,\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, u_r) + \frac{\partial}{\partial \theta}(r\sin\theta\, u_\theta) + \frac{\partial}{\partial \phi}(r u_\phi)\right\},$$

$$\text{curl}\,\mathbf{u} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(u_\phi\sin\theta) - \frac{\partial u_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r}(r u_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.$$

$$\tau_{rr} = \lambda \text{div}\,\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}.$$

5. A finite cylinder of length l is made of a linearly elastic material of density ρ; Lamé constants λ and μ; and undeformed radius 1. The cylinder rotates at a constant angular velocity $\omega > 0$ about its axis which means that in a frame rotating with the cylinder, the governing equations are the Navier–Lamé equations

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} + \boldsymbol{F} = \mathbf{0},$$

where $\boldsymbol{F} = \rho\omega^2 r\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}$ is a radial unit vector directed away from the axis of the cylinder. The ends of the cylinder are subjected to a pressure P.

(i) Assuming that the resulting displacements can be written in the form

$$\boldsymbol{u} = u_r(r)\,\hat{\mathbf{r}} + u_z(z)\,\hat{\mathbf{z}},$$

where (r, θ, z) are cylindrical polar coordinates and $\hat{\mathbf{z}}$ is a unit vector directed along the axis of the cylinder, solve the Navier–Lamé equations to find the general solution for the displacement field throughout the cylinder.

(ii) Write down the stress boundary condition at the outer boundary $r = 1$.

(iii) Find the stress tensor, τ_{ij}, and use the stress boundary condition to find an explicit expression for u_r.

(iv) Find the external pressure that must be exerted on the ends of the cylinder to prevent it from contracting axially and explain why it is not zero.

[20 marks]

You may use the results that in cylindrical polar coordinates (r, θ, z)

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div} \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl} \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

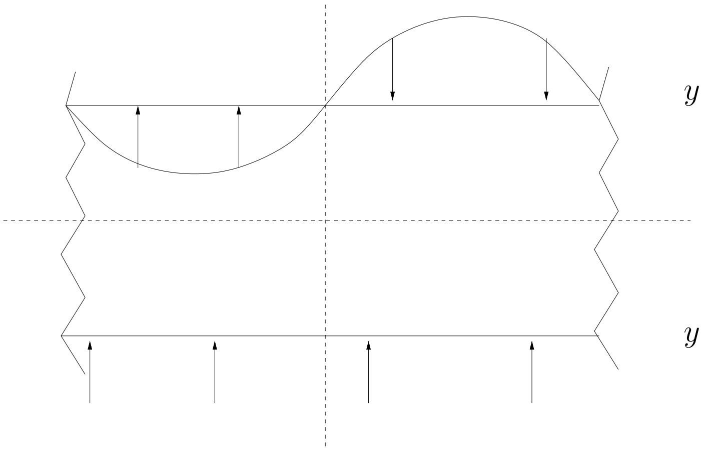
$$\tau_{rr} = \lambda \operatorname{div} \mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda \operatorname{div} \mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda \operatorname{div} \mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

6. An infinite strip of linearly elastic material is subjected to sinusoidal pressure loading on its upper surface and constant pressure loading on the lower surface so that

$$\tau_{xy} = 0, \tau_{yy} = -\beta \sin x \quad \text{on } y = h,$$

$$\tau_{xy} = 0, \tau_{yy} = -\alpha \quad \text{on } y = -h.$$



(i) By considering the boundary conditions for τ_{yy}, explain why the ansatz $\Phi(x, y) = \sin x f(y) + \frac{1}{2}x^2 g(y)$ is a plausible form for the Airy stress function. Hence, if the ansatz is assumed, show that f and g must be of the forms

$$f(y) = A \cosh y + B \sinh y + Cy \cosh y + Dy \sinh y \quad \text{and} \quad g(y) = ay + b,$$

where A, B, C, D, a and b are constants.

(ii) Explain why the boundary conditions at the upper and lower surfaces of the strip cannot be satisfied exactly using the Airy stress function Φ, but that they can be satisfied in a Saint-Venant sense.

(iii) Use the boundary conditions at the upper and lower surfaces of the strip to find the unknown constants, A, B, C,D, a, b for the Saint-Venant solution.

(iv) Under what conditions is the Saint-Venant solution expected to be a good approximation to the true solution.

[20 marks]

You may use the results that for an Airy stress function $\Phi(x_1, x_2)$,

$$\nabla^4 \Phi = \frac{\partial^4 \Phi}{\partial x_1^4} + 2\frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} = 0,$$

and

$$\tau_{11} = \frac{\partial^2 \Phi}{\partial x_2^2}, \quad \tau_{22} = \frac{\partial^2 \Phi}{\partial x_1^2}, \quad \tau_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}.$$

[20 marks]

END OF EXAMINATION PAPER