Two hours

THE UNIVERSITY OF MANCHESTER

ELASTICITY

20 January, 2012

14:00 – 16:00

Answer **ALL** questions in Section A and **ALL** questions in Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer all **FOUR** questions

A1. A two-dimensional elastic body has an undeformed configuration given by a triangle defined by the vertices $(x_1, x_2) = (0, 0)$, $(1, 0)$ and $(0, 1)$. The body is subjected to the displacement field

$$u_1 = \epsilon\,(2x_2 + 4x_1), \qquad u_2 = \epsilon\, x_1,$$

where ϵ is a small positive constant.

(i) By considering the displacement of the vertices, or otherwise, sketch the deformed triangle.

(ii) Determine the strain tensor e_{ij} and the rotation tensor ω_{ij} corresponding to this displacement field.

(iii) Find the principal strains and principal axes of strain throughout the body.

[12 marks]

A2. A three-dimensional linear elastic body has a constitutive law given by

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

where λ and μ are the Lamé coefficients. The body is loaded such that throughout the body the stress tensor is given by $\tau_{11} = 2$, $\tau_{22} = 1$, $\tau_{33} = 2$ and $\tau_{ij} = 0$ when $i \neq j$.

(i) Using the constitutive law derive the strain tensor, e_{ij}, corresponding to the stress field described above.

(ii) Hence, or otherwise, reconstruct the displacement field of the body in the case when $\lambda = \mu = 1$. (You need not derive the most general solution and should suppress rigid body motions when appropriate.)

[10 marks]

A3. A semi-infinite linearly elastic body ($x \leq 0$, and $-\infty < y < \infty$) is loaded only by a surface traction $\boldsymbol{t} = (1, y)$ at $x = 0$. Find values of the coefficients A, B and C to ensure that

$$\phi(x, y) = Axy^2 + By^2 + Cy^4,$$

is a valid Airy stress function that describes the stress field within the elastic body.

[6 marks]

You may use the results that

$$\tau_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \tau_{yy} = \frac{\partial^2 \phi}{\partial x^2} \quad \text{and} \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y},$$

without proof.

A4. An infinite linear elastic body with Lamé coefficients λ and μ contains a spherical inclusion of undeformed radius a. The inclusion has an internal pressure given by $p = p_0$.

(i) Explain why the displacement field in the elastic body will be of the form

$$\boldsymbol{u} = \frac{K}{r^2}\,\hat{\mathbf{r}},$$

where K is a constant; r is the distance from the centre of the inclusion; and $\hat{\mathbf{r}}$ is a unit vector directed away from the centre of the inclusion. You should state any assumptions clearly and may assume that the governing equations within the elastic body are the Navier–Lamé equations

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = \mathbf{0}.$$

(ii) Explain why the boundary condition on the elastic solid is given by

$$\tau_{rr}|_{r=a} = -p_0.$$

(iii) Hence, find the value of the unknown constant K and the radius of the deformed inclusion in terms of λ, μ, p_0 and a.

[12 marks]

You may use the results that in spherical polar coordinates (r, θ, ϕ)

$$\text{grad}\, f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}},$$

$$\text{div}\,\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, u_r) + \frac{\partial}{\partial \theta}(r\sin\theta\, u_\theta) + \frac{\partial}{\partial \phi}(r u_\phi)\right\},$$

$$\text{curl}\,\mathbf{u} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(u_\phi \sin\theta) - \frac{\partial u_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r}(r u_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.$$

$$\tau_{rr} = \lambda \text{div}\,\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}.$$

SECTION B

Answer **BOTH** questions

B5. A cylindrical tube is made of a homogeneous, isotropic, linearly elastic material with Lamé constants λ and μ. The undeformed inner radius of the tube is a and the undeformed outer radius is b, where $b > a$. The cylinder is subject to an internal pressure p_a at $r = a$ and is not loaded externally.

(i) Assuming that the resulting displacements depend only on the distance from the axis of the cylinder and act only in the radial direction, solve the the Navier–Lamé equations

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = \mathbf{0},$$

to find the displacement field throughout the cylinder.

(ii) Find the stress tensor, τ_{ij}, throughout the body and explain why τ_{zz} is not zero.

(iii) Consider now the case when $\mu = 0$. If the cylinder ruptures when $\frac{1}{2}\left(\tau_{ij}\tau_{ij} - \frac{1}{3}\tau_{ii}\tau_{jj}\right) > 4$ anywhere within the body, find the maximum internal pressure that can be supported without rupture.

[20 marks]

You may use the results that in cylindrical polar coordinates (r, θ, z)

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div} \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl} \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

$$\tau_{rr} = \lambda \operatorname{div} \mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda \operatorname{div} \mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda \operatorname{div} \mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

B6. A semi-infinite elastic plate ($-\infty < x < \infty, 0 \leq y < \infty$) is loaded by a surface traction

$$\tau_{yy} = \sum_{n=1}^{\infty} \frac{1}{n} \cos(n\pi x), \quad \text{and} \quad \tau_{xy} = 0, \quad \text{at} \quad y = 0.$$

(i) State the equation that must be satisfied by an Airy stress function Φ.

(ii) By considering a sum of separable Airy stress functions of the form

$$\Phi_n(x, y) = f_n(x)\, g_n(y),$$

find an Airy stress function that describes the stress field throughout the elastic plate, assuming that the stresses are everywhere finite.

[20 marks]

END OF EXAMINATION PAPER