Two hours

UNIVERSITY OF MANCHESTER

ELASTICITY

25 January, 2011

14:00 – 16:00

Answer **ALL** questions in Section A and **ALL** questions in Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer all **FOUR** questions

A1. The displacement field in a two-dimensional elastic body whose undeformed configuration is the unit square, $x_1 \in [0,1]$ and $x_2 \in [0,1]$, is given by

$$u_1 = \epsilon \left[15 + \frac{1}{2}x_1^2 - \frac{1}{3}x_1^3\right], \quad \text{and} \quad u_2 = \epsilon \left[\frac{1}{4}x_2 - \frac{1}{3}\left(x_2 - \frac{1}{3}\right)^3\right],$$

where ϵ is a small positive constant.

(i) Determine the strain tensor e_{ij} and the rotation tensor ω_{ij} corresponding to this displacement field. Is the deformation homogeneous?

(ii) Find the principal strains and principal axes of strain throughout the body. Hence, describe the deformation.

(iii) Find the maximum value of the sum of principal strains within the body and the position at which it occurs.

[12 marks]

A2.

(i) Prove that if α, T and N are constants

$$\phi(x,y) = \frac{1}{\alpha^2}\left[N\mathrm{e}^{\alpha y} + \alpha(T-N)y\mathrm{e}^{\alpha y}\right]\sin(\alpha x),$$

is a valid Airy stress function.

(ii) If ϕ represents the solution for the stress field in the semi-infinite plate $(-\infty < x < \infty, -\infty < y \leq 0)$, find the corresponding tractions that must be applied at the surface $y = 0$. Give a physical interpretation of the constants T and N.

[10 marks]

You may use the results that

$$\tau_{xx} = \frac{\partial^2\phi}{\partial y^2}, \quad \tau_{yy} = \frac{\partial^2\phi}{\partial x^2} \quad \text{and} \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x \partial y},$$

without proof.

A3. A three-dimensional linearly elastic body with Lamé moduli λ and μ is in a state of plane stress such that all stresses are independent of x_3 and act only in planes parallel to $x_3 = 0$.

(i) Write down the form of the stress tensor τ_{ij} within the body indicating clearly any components that are zero.

(ii) Rearrange the constitutive equation

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},$$

to find explicit expressions for the strain e_{ij} as a function of the stress τ_{ij}.

(iii) Hence write down the form of the strain tensor for the body. Is the body necessarily also in a state of plane strain?

[8 marks]

A4. A linearly-elastic sphere has undeformed radius a and Lamé constants λ and μ. The sphere's surface is uniformly glued to the inside of a rigid sphere of the same diameter which is then rotated through an angle Ω about an axis passing through the centre of the sphere.

(i) Explain why the displacement boundary condition at the outer wall of the sphere is

$$\boldsymbol{u}(r = a) = a\Omega \sin\theta\, \hat{\boldsymbol{\phi}},$$

where the spherical polar coordinate system (r, θ, ϕ) is defined so that the axis of rotation is formed by the lines $\theta = 0$ and $\theta = \pi$ and $\hat{\boldsymbol{\phi}}$ is a unit vector in the azimuthal direction.

(ii) By assuming a displacement field of the form $\boldsymbol{u} = f(r)\,\Omega\,\sin\theta\,\hat{\boldsymbol{\phi}}$, show that the Navier–Lamé equations in the absence of any body forces:

$$(\lambda + 2\mu)\,\mathrm{grad}\,(\mathrm{div}\,\boldsymbol{u}) - \mu\,\mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{u}) = \mathbf{0}$$

reduce to the ordinary differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big[rf(r)\Big] - \frac{2f(r)}{r} = 0, \tag{1}$$

(iii) Solve equation (1), subject to suitable boundary conditions, to find the displacement field within the sphere.

(iv) Explain why the resulting displacement will not induce any stresses.

[10 marks]

You may use the results that in spherical polar coordinates (r, θ, ϕ)

$$\mathrm{grad}\, f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}},$$

$$\mathrm{div}\,\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, u_r) + \frac{\partial}{\partial\theta}(r\sin\theta\, u_\theta) + \frac{\partial}{\partial\phi}(ru_\phi)\right\},$$

$$\mathrm{curl}\,\mathbf{u} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\phi\sin\theta) - \frac{\partial u_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(ru_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}.$$

SECTION B

Answer **BOTH** questions

B5. A cylindrical tube is made of a homogeneous, isotropic, linearly-elastic material with Lamé constants λ and μ. The undeformed inner radius of the tube is a and the undeformed outer radius is b, where $b > a$. The cylinder is subject to an internal pressure p_a at $r = a$ and external pressure p_b at $r = b$.

(i) Assuming that the resulting displacements depend only on the distance from the axis of the cylinder and act only in the radial direction, show that the general solution of the Navier–Lamé equations

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0},$$

is

$$\boldsymbol{u} = \left(M\,r + \frac{N}{r}\right)\hat{\boldsymbol{r}},$$

where $\hat{\boldsymbol{r}}$ is a unit vector directed away from the axis of the tube and M and N are constants.

(ii) Write down the appropriate stress boundary conditions at $r = a$ and $r = b$ and use them to find the displacement field within the elastic body.

(iii) Show that a positive displacement of the outer boundary is only possible if $p_a > Cp_b$, where C is a constant to be found. What is the limiting value of C as $b \to a$?

[20 marks]

You may use the results that in cylindrical polar coordinates (r, θ, z)

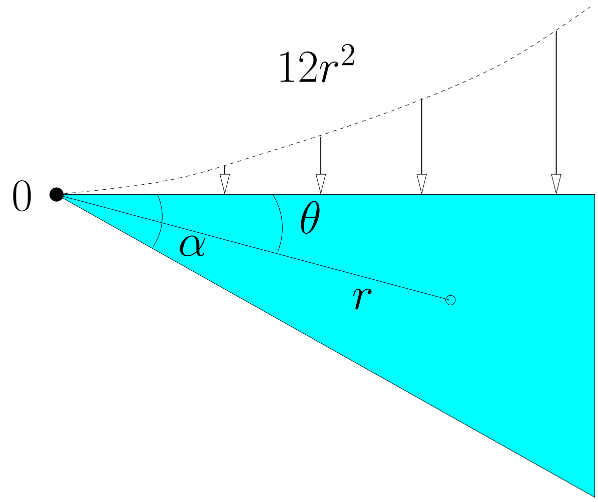
$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial (ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl}\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

$$\tau_{rr} = \lambda \operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda \operatorname{div}\mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda \operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

B6. An semi-infinite wedge of angle α is loaded on one surface by a pressure that increases quadratically with distance from its apex. The wedge is composed of a linearly elastic material. A polar coordinate system is defined with its origin at the apex of the wedge such that the loaded surface is at $\theta = 0$ and the other surface is at $\theta = \alpha$ and is not loaded. The pressure on the upper face is given by $12r^2$.



(i) State the equation that must be satisfied by an Airy stress function Φ.

(ii) Verify that

$$\Phi(r, \theta) = r^4 \left[A \cos 4\theta + B \sin 4\theta + C \cos 2\theta + D \sin 2\theta\right],$$

is a valid Airy stress function, where (r, θ) is the polar coordinate system introduced above.

(iii) By applying continuity of stress boundary conditions at the faces $\theta = 0$ and $\theta = \alpha$, find four simultaneous equations that relate the constants A, B, C and D.

(iv) Solve these equations when $\alpha = \pi/4$ and, hence, find the stress field throughout the body in this case.

[20 marks]

You may use the results that in plane polar coordinates:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad 2e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

$$\nabla^2 f(r, \theta) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2},$$

and for an Airy stress function, $\Phi(r, \theta)$

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}, \quad \tau_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta}.$$

END OF EXAMINATION PAPER