Two hours

UNIVERSITY OF MANCHESTER

ELASTICITY

21 January, 2010

09:45 – 11:45

Answer **ALL** questions in Section A and **ALL** questions in Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer all **FOUR** questions

A1. The displacement field in a two-dimensional elastic body is given by

$$\begin{aligned} u_1 &= \epsilon\ [7 + 2(x_2 - 1)(x_1 - 1)]\,, \\ u_2 &= \epsilon\ \left[\frac{1}{2}x_2^2\right], \end{aligned}$$

where ϵ is a small positive constant.

(i) Determine the strain tensor e_{ij} and the rotation tensor ω_{ij} corresponding to displacement field.

(ii) For any undeformed configuration, explain why there are no points at which the body is unstrained.

(iii) Find the principal strains and principal axes of strain at the points at which the rotation tensor is zero and, hence, describe the deformation at these points.

[12 marks]

A2.

(i) Explain why it is not possible to recover a valid two-dimensional displacement field from the strain tensor given by

$$e_{11} = x_2^3, \quad e_{12} = 0, \quad e_{22} = 0.$$

(ii) Find a function $f(x_1, x_2)$ that enables a valid two-dimensional displacement field to be found from the strain tensor

$$e_{11} = x_2^3, \quad e_{12} = 0, \quad e_{22} = f(x_1, x_2).$$

and calculate one such displacement field.

[10 marks]

You may make use the strain compatibility equations

$$e_{ij,kl} + e_{kl,ij} - e_{kj,il} - e_{il,kj} = 0.$$

without proof.

A3. The constitutive equation for a homogeneous, isotropic linearly elastic material is given by

$$\tau_{ij} = \tau_{ij}^0 + \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}, \tag{1}$$

where τ_{ij} is the stress tensor, τ_{ij}^0 is a pre-stress, λ and μ are the Lamé constants, e_{ij} is the strain tensor and δ_{ij} is the Kronecker delta.

(i) Rearrange equation (1) to find an explicit formula for the strain as a function of the stress.

(ii) Hence, or otherwise, find an expression for the strain when the stress τ_{ij} is zero. Explain what happens to your expression when there is no pre-stress.

[8 marks]

A4. A linearly-elastic cylinder of infinite extent has undeformed radius a and Lamé constants λ and μ. The cylinder is forced inside a rigid tube of smaller radius $b < a$ in such a way that the cylinder wall is not twisted.

(i) Write down the displacement boundary conditions at the outer wall of the cylinder.

(ii) Show that a displacement field of the form $\boldsymbol{u} = (u_r, u_\theta, u_z) = f(r)\hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is a unit vector directed away from the cylinder's axis, can satisfy the boundary conditions. Furthermore, show that the Navier–Lamé equations in the absence of any body forces:

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \mu\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u} = \mathbf{0}$$

reduce to the ordinary differential equation

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rf(r)\right)\right] = 0. \tag{2}$$

(iii) Solve equation (2) to find the displacement field within the cylinder.

(iv) Find the normal stress exerted by the rigid pipe at the outer boundary of the cylinder.

[10 marks]

You may use the results that in cylindrical polar coordinates (r, θ, z)

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl}\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda\operatorname{div}\mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

SECTION B

Answer **BOTH** questions

B5. An hollow sphere is made of a homogeneous, isotropic, linearly-elastic material with Lamé constants λ and μ. The undeformed inner radius of the sphere is a and the undeformed outer radius is b. The sphere is subject to an internal pressure p_a at $r = a$ and external pressure p_b at $r = b$, where $b > a$.

(i) Assuming that the resulting displacements depend only on the distance from the centre of the sphere and act only in the radial direction, show that the general solution of the Navier–Lamé equations

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0},$$

is

$$\boldsymbol{u} = \left(C\,\frac{r}{3} + \frac{D}{r^2} \right) \hat{\boldsymbol{r}},$$

where $\boldsymbol{r}$ is a unit vector directed away from the centre of the sphere and C and D are constants.

(ii) Write down the appropriate stress boundary conditions at $r = a$ and $r = b$ and use them to find the displacement field within the elastic body.

(iii) Find the displacement field when the internal and external pressures are both equal to a constant value p and explain why it is not zero when $p > 0$.

(iv) Show that a positive displacement at the outer boundary is only possible if $p_a > K\,p_b$, where $K > 1$ is a constant to be found.

[20 marks]

You may use the results that in spherical polar coordinates (r, θ, ϕ)

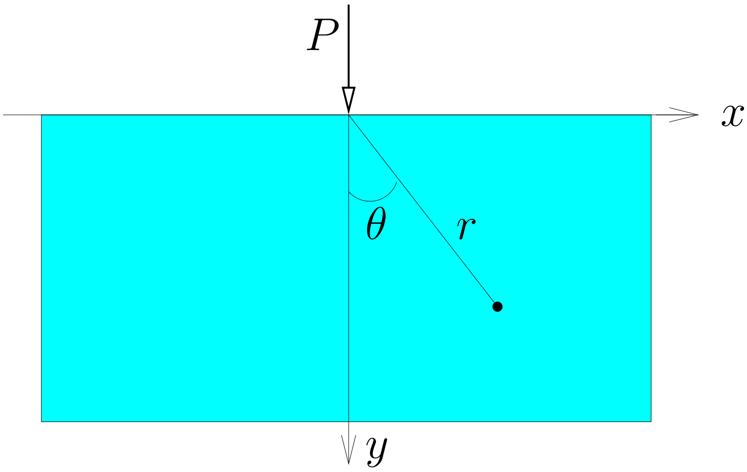
$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}},$$

$$\operatorname{div} \mathbf{u} = \frac{1}{r^2 \sin\theta}\left\{ \frac{\partial}{\partial r}(r^2 \sin\theta\, u_r) + \frac{\partial}{\partial \theta}(r \sin\theta\, u_\theta) + \frac{\partial}{\partial \phi}(r u_\phi) \right\},$$

$$\operatorname{curl} \mathbf{u} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ u_r & r u_\theta & r \sin\phi\, u_\phi \end{vmatrix}.$$

$$\tau_{rr} = \lambda \operatorname{div}\, \boldsymbol{u} + 2\mu\frac{\partial u_r}{\partial r}.$$

B6. A point load of magnitude P is applied to a semi-infinite two-dimensional elastic body occupying the region $y \geq 0$. The load acts only in the y-direction and is applied at the origin of the Cartesian coordinate system.



(i) State the equation that must be satisfied by an Airy stress function Φ.

(ii) Verify that

$$\Phi(r,\theta) = -\frac{P}{\pi} r\,\theta\,\sin\theta,$$

is a valid Airy stress function, where (r,θ) is a polar coordinate system sketched above.

(iii) Calculate the corresponding stress components τ_{rr}, $\tau_{r\theta}$ and $\tau_{\theta\theta}$.

(iv) By calculating the resultant force exerted in the x- and y-directions on a semi-circle a distance r from the point of application of the load, show that the Airy stress function represents the solution of the problem described above.

(v) Find the resulting displacement field assuming that the body is in a state of plane stress in which case

$$Ee_{rr} = \tau_{rr} - \nu\tau_{\theta\theta}, \quad Ee_{\theta\theta} = \tau_{\theta\theta} - \nu\tau_{rr}, \quad Ee_{r\theta} = (1+\nu)\tau_{r\theta},$$

where E is the Young's modulus and ν is the Poisson ratio of the material.

[20 marks]

You may use the results that in plane polar coordinates:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad 2e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

$$\nabla^2 f(r,\theta) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2},$$

and for an Airy stress function, $\Phi(r,\theta)$

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}, \quad \tau_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta}.$$

END OF EXAMINATION PAPER