Two hours

UNIVERSITY OF MANCHESTER

ELASTICITY

Wednesday 21st January, 2009

09.45 - 11.45

Answer **ALL** questions in Section A and **ALL** question in Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer all **THREE** questions

A1. The displacement field in a two-dimensional elastic body is given by

$$\begin{aligned} u_1 &= \epsilon\ \left[5 + (x_2 - 2)^2\right], \\ u_2 &= \epsilon\ \left[3 + (x_2 - 2)(x_1 - 1)\right], \end{aligned}$$

where ϵ is a small positive constant.

(i) Determine the strain tensor e_{ij} and the rotation tensor ω_{ij} of this displacement field.

(ii) Find the principal strains as functions of position (x_1, x_2).

(iii) If the undeformed body is the unit square $0 \le x_1 \le 1$ and $0 \le x_2 \le 1$, find the point, P, at which the maximum strain occurs within the body. What is the value of this maximum strain?

[10 marks]

A2.

(i) Prove that

$$\phi(x, y) = \frac{3A}{4l}\left(\frac{x^3 y}{3l^2} - xy\right) + \frac{B}{4l}x^2,$$

is a valid Airy stress function.

(ii) Assuming that ϕ describes the stress in a block occupying the region $-l \le x \le l$, $-h \le y \le 0$, calculate the stress field throughout the block.

(iii) Determine the resultant force acting on the upper face of the block ($y = 0$) and hence give a physical meaning to the constants A and B.

[12 marks]

A3. A three-dimensional elastic body is in a state of plane strain such that all deformations are independent of x_3 and lie in planes parallel to $x_3 = 0$. The body is linearly elastic with Lamé moduli λ and μ.

(i) Write down the form of the strain tensor e_{ij} within the body. In particular, indicate clearly any components that are zero.

(ii) Use the constitutive law

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

to deduce the corresponding form of the stress tensor in terms of the non-zero components of the strain tensor.

(iii) Can the stresses act in the x_3 direction? Justify your answer.

[8 marks]

A4. A linearly-elastic circular cylinder with Lamé modulii λ and μ has undeformed radius a, undeformed height 1 and occupies the domain $0 \le r \le a$ and $0 \le z \le 1$ in a cylindrical polar coordinate system (r, θ, z). The cylinder undergoes a purely torsional deformation such that

$$\boldsymbol{X} = \boldsymbol{x} + Cz\,\hat{\boldsymbol{\theta}}, \quad \text{on the boundary} \quad r = a,$$

where $\boldsymbol{X}$ is the deformed position, $\boldsymbol{x}$ is the undeformed position, C is a constant, and $\hat{\boldsymbol{\theta}}$ is a unit vector in the θ direction.

(i) Show that a displacement field of the form $\boldsymbol{u} = Crz/a\ \hat{\boldsymbol{\theta}}$ satisfies the boundary conditions at $r = a$ and also satisfies the steady Navier–Lamé equations in the absence of any body forces:

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}.$$

(ii) Find all components of the stress tensor that corresponds to this displacement field.

(ii) Calculate the total resultant force, $\boldsymbol{F}$, and resultant torque about the cylinder's axis, $\boldsymbol{T}$, on upper surface of the cylinder $z = 1$:

$$\boldsymbol{F} = \int_S \boldsymbol{t}\, \mathrm{d}S \quad \text{and} \quad \boldsymbol{T} = \int_S \boldsymbol{r} \times \boldsymbol{t}\, \mathrm{d}S,$$

where $\boldsymbol{r}$ is the distance to the axis of rotation and $\boldsymbol{t}$ is the traction on the surface.

[10 marks]

You may use the results that in cylindrical polar coordinates (r, θ, z)

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div} \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl} \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

$$\tau_{rr} = \lambda \operatorname{div} \mathbf{u} + 2\mu \frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda \operatorname{div} \mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda \operatorname{div} \mathbf{u} + 2\mu \frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

SECTION B

Answer **ALL** two questions

B5. An solid sphere of an isotropic, homogeneous elastic material with Lamé moduli λ and μ has undeformed radius a and is subjected to body force given by

$$\boldsymbol{F} = -G\boldsymbol{x},$$

where G is a constant and $\boldsymbol{x}$ is the position vector to the centre of the sphere. The surface of the sphere is subjected to a constant external pressure P.

(i) Assuming that the displacement field has the form $\boldsymbol{u} = \boldsymbol{x} f(r)$, where $r = |\boldsymbol{x}|$ show that

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{G}{2\mu + \lambda} = 0.$$

(ii) Hence, or otherwise, find f and show that the radial displacement at the surface of the sphere is given by

$$-\frac{Ga^3 + Pa}{5(3\lambda + 2\mu)}.$$

[20 marks]

You may use the results that in spherical polar coordinates (r, θ, ϕ)

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}},$$

$$\operatorname{div} \mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, u_r) + \frac{\partial}{\partial \theta}(r\sin\theta\, u_\theta) + \frac{\partial}{\partial \phi}(r u_\phi)\right\},$$

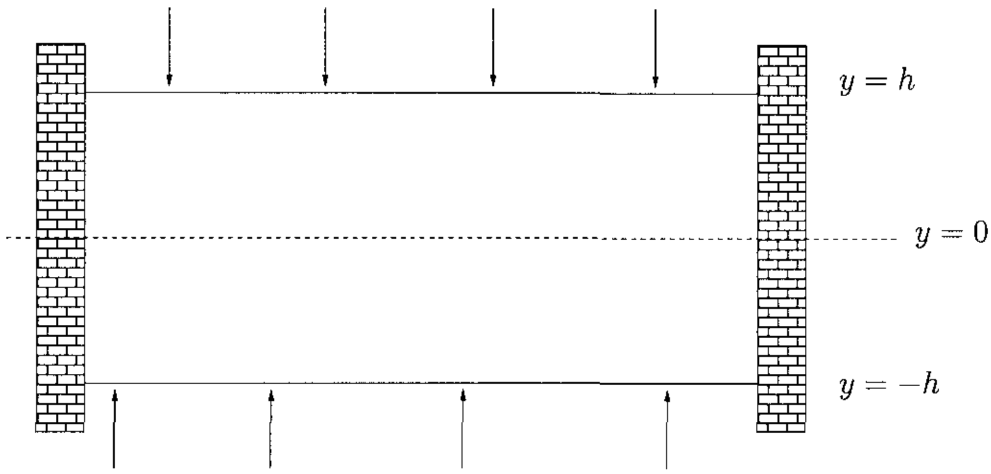
$$\operatorname{curl} \mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ u_r & r u_\theta & r\sin\phi\, u_\phi \end{vmatrix}.$$

$$\tau_{rr} = \lambda \operatorname{div} \boldsymbol{u} + 2\mu\frac{\partial u_r}{\partial r}.$$

B6. A rectangular beam of a homogeneous, isotropic, linearly-elastic material is supported at both ends and subjected to normal sinusoidal loading on its upper and lower surfaces:

$$\tau_{xy} = 0, \tau_{yy} = -\beta \sin x \quad \text{on } y = h,$$

$$\tau_{xy} = 0, \tau_{yy} = -\alpha \sin x \quad \text{on } y = -h.$$



(i) By assuming that the Airy stress function has the form $\Phi(x, y) = \sin x f(y)$ show that f must be of the form

$$f(y) = A \cosh y + B \sinh y + Cy \cosh y + Dy \sinh y.$$

(ii) Use the boundary conditions at the upper and lower surfaces of the beam to find the unknown constants, A, B, C and D.

(iii) In the special case when $\alpha = \beta$, find τ_{xx} along the beam's centreline.

END OF EXAMINATION PAPER