Two hours

UNIVERSITY OF MANCHESTER

ELASTICITY

Friday 18th January, 2008

14:00 – 16:00

Answer **ALL** questions in Section A and **ALL** question in Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer all **THREE** questions

A1. The displacement field in a two-dimensional elastic body is given by

$$\begin{aligned} u_1 &= \epsilon\,\left[1 + (x_1 - 1)(x_2 - 1)\right], \\ u_2 &= \epsilon\,\left[2 + (x_1 - 1)^2\right], \end{aligned}$$

where ϵ is a small positive constant.

(i) Determine the strain tensor e_{ij} and the rotation tensor ω_{ij} of this displacement field.

(ii) Find the coordinates of a point $\boldsymbol{P}$ at which the displacement field does not induce any deformation. State clearly the condition used to specify that there is no deformation.

(iii) Find the principal strains as functions of position (x_1, x_2). What are the values of the principal strains at the point $\boldsymbol{P}$? Find the principal strains and principal axes of strain for all points on the line $\mathbf{r} = (1, y)$.

[12 marks]

A2. An elastic cylinder of constant arbitrary cross-section of area A is in static equilibrium. The axis of the cylinder is in the x_2 direction and its upper and lower surfaces are located at $x_2 = l$ and $x_2 = 0$, respectively. The stress field in the cylinder is given by

$$\tau_{11} = \tau_{33} = -p, \quad \tau_{22} = -\gamma g(l - x_2),$$

$$\tau_{12} = \tau_{23} = \tau_{13} = 0,$$

where g is the acceleration due to gravity and p and γ are constants. Find the surface tractions and resultant forces applied on the upper and lower surfaces of the cylinder. Use the equations of static equilibrium to determine the body force $\boldsymbol{F}$ applied to the body. Hence deduce the resultant force applied on the remaining surface of the cylinder and explain the physical meaning of γ.

[12 marks]

A3. A linearly elastic hollow sphere of internal radius a and outer radius b is subject to an internal pressure p_a and external pressure p_b. Assuming that the displacement field depends only on the radial displacement and acts only in the radial direction $\boldsymbol{u} = (u_r(r), 0, 0)$, use the Navier-Lamé equations

$$(\lambda + 2\mu)\ \text{grad div}\ \mathbf{u} - \mu\ \text{curl curl}\ \mathbf{u} + \mathbf{F} = \mathbf{0}$$

to determine the displacement field. Show that the displacement field can be written in the form

$$u_r(r) = (Ap_a + Bp_b)r + \frac{Cp_a + Dp_b}{r^2}$$

and determine the coefficients A, B, C and D in terms of a, b, λ and μ.

[16 marks]

You may use the results that in spherical polar coordinates (r, θ, ϕ)

$$\text{grad}\, f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}},$$

$$\text{div}\,\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, u_r) + \frac{\partial}{\partial \theta}(r\sin\theta\, u_\theta) + \frac{\partial}{\partial \phi}(ru_\phi)\right\},$$

$$\text{curl}\,\mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ u_r & ru_\theta & r\sin\phi\, u_\phi \end{vmatrix}.$$

$$\tau_{rr} = \lambda\,\text{div}\ \boldsymbol{u} + 2\mu\frac{\partial u_r}{\partial r}.$$

SECTION B

Answer **ALL** two questions

B4. Fig. 1 shows a cantilever beam (length L and height $2H$) which is in a state of plane strain. The beam's upper (horizontal) face (face I) is loaded by a constant pressure q and the faces II and III are not loaded. A St. Venant solution for the stress field in the beam can be found using the ansatz

$$\Phi = ax_1^2 + bx_1^2 x_2 + cx_1^2 x_2^3 + dx_2^5 + \frac{1}{20}\frac{q}{H}x_2^3$$

for the Airy stress function Φ.

(i) State the stress boundary conditions along the three faces I, II and III.

(ii) Determine the stresses τ_{11}, τ_{12} and τ_{22} in the beam in terms of the constants a, b, c and d.

(iii) Determine the constants a, b, c and d from the governing equation and from the stress boundary conditions on the faces I and III.

(iv) Are the boundary conditions on face II fulfilled? Explain why (and in what sense) the solution represents a St. Venant solution to the problem.

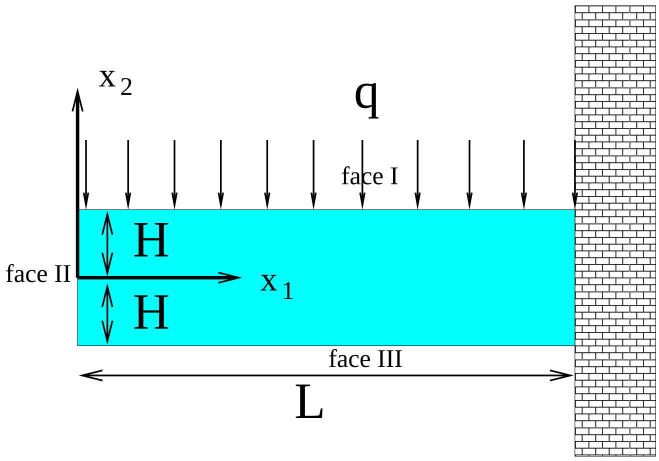

Figure 1: Sketch of a cantilever beam, loaded by a constant pressure q on its upper face.

[20 marks]

You may use the results that for an Airy stress function $\Phi(x_1, x_2)$,

$$\nabla^4\Phi = \frac{\partial^4\Phi}{\partial x_1^4} + 2\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\Phi}{\partial x_2^4} = 0,$$

and

$$\tau_{11} = \frac{\partial^2\Phi}{\partial x_2^2}, \quad \tau_{22} = \frac{\partial^2\Phi}{\partial x_1^2}, \quad \tau_{12} = -\frac{\partial^2\Phi}{\partial x_1 \partial x_2}.$$

B5. In cylindrical polar coordinates (r, θ, z), the constitutive law for a linearly elastic material is

$$\tau_{ij} = \lambda\delta_{ij}\,(e_{rr} + e_{\theta\theta} + e_{zz}) + 2\mu\, e_{ij},$$

where τ_{ij} is the stress tensor; e_{ij} is the strain tensor; and λ and μ are the Lamé moduli.

(i) Show that

$$\tau_{rr} + \tau_{\theta\theta} + \tau_{zz} = (3\lambda + 2\mu)\,(e_{rr} + e_{\theta\theta} + e_{zz})\,.$$

(ii) Hence, assuming a state of plane stress, $\tau_{zz} = \tau_{zr} = \tau_{z\theta} = 0$, show that

$$e_{rr} = \frac{1}{E}\left(\tau_{rr} - \nu\tau_{\theta\theta}\right), \quad e_{\theta\theta} = \frac{1}{E}\left(\tau_{\theta\theta} - \nu\tau_{rr}\right), \quad e_{r\theta} = \frac{1+\nu}{E}\tau_{r\theta},$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}.$$

(iii) If the Airy stress function $\Phi(r)$ is a function only of r, show that the general solution is

$$\Phi = Ar^2 \ln r + Br^2 + C\ln r + D.$$

and write down the corresponding stresses τ_{rr}, $\tau_{\theta\theta}$, $\tau_{r\theta}$.

(iv) Use the results from part (b) to find expressions for the strains and hence the general solution for the two-dimensional displacement field (u_r, u_θ).

You may use the results that in cylindrical polar coordinates:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad 2e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

$$\nabla^4 f(r) = \frac{1}{r}\left[r\left(\frac{1}{r}\left[r f_{,r}\right]_{,r}\right)_{,r}\right]_{,r}$$

and for an Airy stress function, $\Phi(r, \theta)$

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r}, \quad \tau_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2}, \quad \tau_{r\theta} = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}.$$

[20 marks]

END OF EXAMINATION PAPER