Two Hours

UNIVERSITY OF MANCHESTER

ELASTICITY

2003

Examinationtime not specified

Answer **all** three questions in **SECTION A** (30 marks in total)
and
all of the two questions in **SECTION B** (25 marks each).
The total for the paper is 80 marks. A further 20 marks are available from course work during the semester making a total of 100.
NOTE: All elastic bodies referred to in the following questions are homogeneous, isotropic and linearly elastic.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer all **three** questions

A1. The stress field in a cube ($x_i \in [-1, 1]$) which is loaded by a body force F_i, is given by

$$(\tau_{ij}) = \begin{pmatrix} x_1^2 & 3x_1x_2 & x_1x_2^2 \\ 3x_1x_2 & x_1 + x_2^2 & x_2x_3 \\ x_1x_2^2 & x_2x_3 & -x_2^2x_3 + x_3^2 \end{pmatrix}$$

(i) Determine the components of the body force.
(ii) What is the resultant force acting upon the cube on the surface $x_3 = 1$?

[10 marks]

A2. The displacement field in a two-dimensional elastic body is given by

$$\begin{aligned} u_1 &= \epsilon \left(x_1 + (x_2 - 2)^2 \right), \\ u_2 &= \epsilon \left(x_2 + (x_1 - 1)^2 \right), \end{aligned}$$

where ϵ is a small positive constant.
(i) Determine the strain tensor e_{ij}.
(ii) At the point $\mathbf{r} = (3, 2)$ find the extension $e_{\mathbf{n}}$ of the line element $\mathbf{n}ds$, where $\mathbf{n} = (3/5, 4/5)$.
(iii) Determine the maximum normal strain at $\mathbf{r} = (1, 1)$.

[10 marks]

A3. (i) Show that

$$\phi(x, y) = \frac{3T}{4a}\left(xy - \frac{x^3y}{3a^2} \right) + \frac{N}{4a}x^2$$

is a valid Airy stress function.
(ii) Now assume that ϕ describes the stress in the block occupying the region $-a \leq x \leq a, \ \ -h \leq y \leq 0$, sketched in Fig. 1. Calculate the stress field.
(iii) Determine the resultant force acting on the upper face of the block and hence give a physical meaning to the constants T and N.

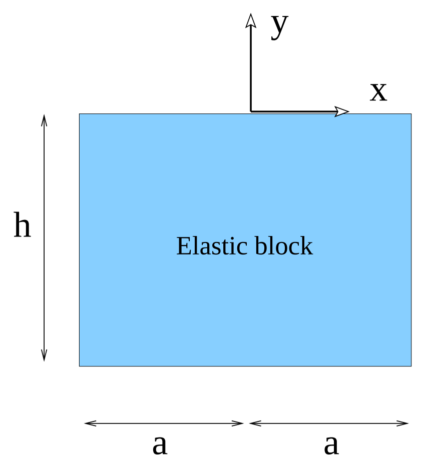

Fig. 1: Sketch of an elastic block.

[10 marks]

SECTION B

Answer **all two** questions

B4.

Fig. 2 shows an elastic half-plane (located in $y \leq 0$), the surface of which is subject to the periodic traction

$$\mathbf{t} = \sigma_0 \cos(x)\ \mathbf{e}_y \quad \text{along } y = 0,$$

and to gravity which acts vertically downwards so that the body force is given by $\mathbf{F} = -\rho g\ \mathbf{e}_y$, where $\mathbf{e}_y$ is the unit vector in the y-direction. You can assume that the material is in a state of plane strain.

(i) Decompose the stress tensor τ_{ij} into two components such that $\tau_{ij} = \tau_{ij}^{(h)} + \tau_{ij}^{(p)}$, where $\tau_{ij}^{(p)}$ balances the body force.

(ii) Use an Airy stress function ϕ to determine $\tau_{ij}^{(h)}$ [Hint: The stress boundary condition suggests a suitable x-dependence for ϕ].

(iii) Determine the points at which τ_{xx} has local extrema and determine its values at these points.

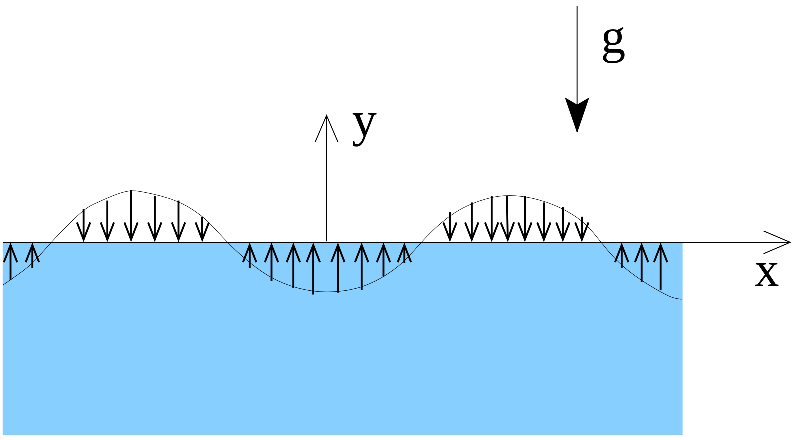

Fig. 2: Sketch of an elastic half-plane subject to gravity and a periodic surface traction.

[25 marks]

B5. The infinitely long circular cylindrical pipe (inner radius a, outer radius b) sketched in Fig. 3 is subject to uniform circumferential shear tractions on its outer and inner surfaces such that the applied traction vectors are $\mathbf{t}|_{r=a} = T\hat{\boldsymbol{\theta}}$ and $\mathbf{t}|_{r=b} = S\hat{\boldsymbol{\theta}}$. There are no body forces and the displacement field can be assumed to be of the form $\mathbf{u} = u(r)\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ is the unit vector in the circumferential direction.

(i) Show that overall equilibrium requires that $S = -\left(\frac{a}{b}\right)^2 T$.

(ii) Determine the displacement and stress fields in the cylinder.

(iii) Explain why the solution for the displacement field is not unique.

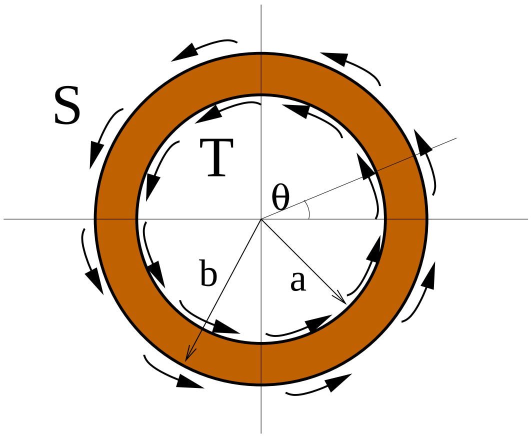

Fig. 3: Sketch of an infinitely long circular cylindrical pipe subject to shear tractions on its inner and outer surfaces.

[25 marks]

Equations in Cylindrical Polar Coordinates

The Navier-Lamé equations in symbolic form are given by

$$(\lambda + 2\mu)\ \text{grad div}\ \mathbf{u} - \mu\ \text{curl curl}\ \mathbf{u} + \mathbf{F} = \mathbf{0}.$$

For $\mathbf{u}(r,\theta,z) = u_r(r,\theta,z)\hat{\mathbf{r}} + u_\theta(r,\theta,z)\hat{\boldsymbol{\theta}} + u_z(r,\theta,z)\hat{\mathbf{z}}$ and $f(r,\theta,z)$ we have

$$\text{grad}\, f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}, \qquad \text{div}\,\mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\text{curl}\,\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ are the basis vectors in the radial, circumferential and axial directions, respectively.