MATH35021: SOLUTION SHEET V[1]

1.) The principal axes of stress are the eigenvectors of the stress tensor, τ_{ij}, *i.e.* the vectors, v_i such that

$$\tau_{ij} v_j = \Lambda v_i, \qquad (1)$$

where Λ is a constant, in fact it is the principal stress.

For a homogeneous, isotropic, linear elastica the constitutive law is

$$\tau_{ij} = \lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij},$$

where e_{ij} is the strain tensor and λ and μ are the Lamé moduli. Using the constitutive law in equation (1) we obtain

$$(\lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij})\; v_j = \Lambda v_i,$$

$$\Rightarrow \lambda\, e_{kk} \delta_{ij}\, v_j + 2\mu\, e_{ij} v_j = \Lambda v_i.$$

Now using the index-switching property of the Kronecker delta and rearranging, we find

$$2\mu\, e_{ij} v_j = (\Lambda - \lambda\, e_{kk})\, v_i \quad \Rightarrow \quad e_{ij} v_j = \frac{\Lambda - \lambda\, e_{kk}}{2\mu}\, v_i.$$

We note that e_{kk} is the dilation and is a strain invariant, so it is a constant, as are λ and μ. Hence, the vectors v_i are also eigenvectors of the strain tensor which means that the principal axes of stress and strain coincide. Moreover, the eigenvalues, or principal strains, P_{strain} are given by

$$P_{\text{strain}} = \frac{P_{\text{stress}} - \lambda\, e_{kk}}{2\mu},$$

because Λ are the principal stresses, P_{stress}.

2.) Here our starting point is the relationship $t_i = \tau_{ij} n_j$. We next use the constitutive law to write

$$t_i = (\lambda\, \delta_{ij}\, e_{kk} + 2\mu\, e_{ij})\, n_j,$$

and the definition of the strain tensor $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, implies that

$$t_i = \lambda\, \delta_{ij} u_{k,k} n_j + \mu\, u_{i,j} n_j + \mu\, u_{j,i} n_j.$$

Using the index-switching property of the Kronecker delta and adding and subtracting the term $\mu\, u_{i,j} n_j$ gives

$$t_i = \lambda\, n_i\, u_{k,k} + 2\mu\, n_j\, u_{i,j} - \mu u_{i,j} n_j + \mu\, u_{j,i} n_j,$$

$$\Rightarrow t_i = \lambda\, n_i\, u_{k,k} + 2\mu\, n_j\, u_{i,j} - 2\mu\, \omega_{ij} n_j,$$

from the definition of the rotation tensor $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$. Finally we convert the expression into dyadic form

$$\boldsymbol{t} = \lambda\, \boldsymbol{n}\; \text{div}\; \boldsymbol{u} + 2\mu\, (\boldsymbol{n} \cdot \boldsymbol{\nabla})\, \boldsymbol{u} - 2\mu\, \boldsymbol{\omega} \times \boldsymbol{n}.$$

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We have used the facts that $u_{k,k} = \text{div}\ \boldsymbol{u}$; that for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $a_j\, b_j = \boldsymbol{a}\cdot\boldsymbol{b}$; and that $\omega_{ij}n_j = \boldsymbol{\omega}\times\boldsymbol{n}$, where $\boldsymbol{\omega} = \frac{1}{2}\text{curl}\ \boldsymbol{u}$. Thus,

$$\boldsymbol{t} = \lambda\,\boldsymbol{n}\ \text{div}\ \boldsymbol{u} + 2\mu\,(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} - \mu\,\text{curl}\ \boldsymbol{u}\times\boldsymbol{n}.$$

The final result is obtained by using the anti-symmetric property of the cross product

$$\boldsymbol{t} = \lambda\,\boldsymbol{n}\ \text{div}\ \boldsymbol{u} + 2\mu\,(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + \mu\,\boldsymbol{n}\times\text{curl}\ \boldsymbol{u}.$$

3.) This is a straightforward exercise in manipulation using the definitions

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \text{and} \quad \nu = \frac{\lambda}{2(\lambda+\mu)},$$

or, more usefully,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

So,

$$\tau_{ij} = \lambda\,\delta_{ij}\,e_{kk} + 2\mu\,e_{ij},$$

give

$$\tau_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\,\delta_{ij}\,e_{kk} + \frac{2E}{2(1+\nu)}e_{ij},$$

$$\Rightarrow \tau_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu}\,\delta_{ij}\,e_{kk}\right).$$

We now invert to find the relationship for the strains in terms of the stresses

$$\frac{1+\nu}{E}\tau_{ij} = e_{ij} + \frac{\nu}{1-2\nu}\,\delta_{ij}\,e_{kk}.$$

If we put $i = j$ in the above, we find

$$\frac{1+\nu}{E}\tau_{ii} = e_{ii} + \frac{3\nu}{1-2\nu}\,e_{kk} = \frac{1-2\nu+3\nu}{1-2\nu}e_{ii} = \frac{1+\nu}{1-2\nu}e_{ii}.$$

$$\Rightarrow e_{kk} = \frac{1-2\nu}{E}\tau_{kk},$$

and so

$$\frac{1+\nu}{E}\tau_{ij} = e_{ij} + \frac{\nu}{1-2\nu}\,\delta_{ij}\,\frac{1-2\nu}{E}\tau_{kk}.$$

Then

$$e_{ij} = \frac{1}{E}\bigg((1+\nu)\tau_{ij} - \nu\,\delta_{ij}\,\tau_{kk}\bigg).$$