MATH35021: SOLUTION SHEET IV[1]

1.) a) The equations of equilibrium are $\tau_{ij,j} + F_i = 0$ or written in component form:

$$\begin{aligned} i=1: &\qquad \tau_{11,1} + \tau_{12,2} + \tau_{13,3} + F_1 = 0,\\ i=2: &\qquad \tau_{21,1} + \tau_{22,2} + \tau_{23,3} + F_2 = 0,\\ i=3: &\qquad \tau_{31,1} + \tau_{32,2} + \tau_{33,3} + F_3 = 0. \end{aligned}$$

We know that $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = 0$ and $\tau_{13} = \tau_{31} = -ax_2$, $\tau_{23} = \tau_{32} = ax_1$ and $\tau_{33} = bx_3$. Thus, the equilibrium equations reduce to

$$F_1 = 0, \quad F_2 = 0, \quad \text{and} \quad b + F_3 = 0.$$

Hence, the body force per unit volume is given by

$$\boldsymbol{F} = \begin{pmatrix} 0 \\ 0 \\ -b \end{pmatrix},$$

and is spatially constant.

b) The surface tractions can be found by using the formula $t_i = \tau_{ij} n_j$, where $\boldsymbol{n}$ is the outer unit normal to the surface in question. It is most convenient to work in cylindrical polar coordinates (r, θ, z) in which case the stress tensor becomes

$$\tau_{ij} = \begin{pmatrix} 0 & 0 & -ax_2 \\ 0 & 0 & ax_1 \\ -ax_2 & ax_1 & bx_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -ar\sin\theta \\ 0 & 0 & ar\cos\theta \\ -ar\sin\theta & ar\cos\theta & bz \end{pmatrix}$$

Top: $z = 0$
In this case, $\boldsymbol{n} = (0, 0, 1)$. Hence,

$$\boldsymbol{t} = \begin{pmatrix} 0 & 0 & -ar\sin\theta \\ 0 & 0 & ar\cos\theta \\ -ar\sin\theta & ar\cos\theta & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -ar\sin\theta \\ ar\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -ax_2 \\ ax_1 \\ 0 \end{pmatrix}.$$

Bottom: $z = -L$
In this case, $\boldsymbol{n} = (0, 0, -1)$. Hence,

$$\boldsymbol{t} = \begin{pmatrix} 0 & 0 & -ar\sin\theta \\ 0 & 0 & ar\cos\theta \\ -ar\sin\theta & ar\cos\theta & -bL \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} ar\sin\theta \\ -ar\cos\theta \\ bL \end{pmatrix} = \begin{pmatrix} ax_2 \\ -ax_1 \\ bL \end{pmatrix}.$$

Curved side: $x_1^2 + x_2^2 = R^2$**,** $r = R$
In this case, $\boldsymbol{n} = (\cos\theta, \sin\theta, 0)$. Hence,

$$\boldsymbol{t} = \begin{pmatrix} 0 & 0 & -aR\sin\theta \\ 0 & 0 & aR\cos\theta \\ -aR\sin\theta & aR\cos\theta & -bz \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

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c) The resultant forces are found by integrating all tractions over each face and the body force over the volume of the body:

$$T_i = \iint_A t_i \mathrm{d}A, \quad \text{Resultant load on face } A,$$

$$B_i = \iiint_V F_i \mathrm{d}V, \quad \text{Resultant body force on volume } V.$$

Considering each face in turn, we have

$$\boldsymbol{T}_{\text{top}} = \int_0^{2\pi}\int_0^R \begin{pmatrix} -ar\sin\theta \\ ar\cos\theta \\ 0 \end{pmatrix} r\mathrm{d}r\mathrm{d}\theta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\boldsymbol{T}_{\text{bottom}} = \int_0^{2\pi}\int_0^R \begin{pmatrix} ar\sin\theta \\ -ar\cos\theta \\ bL \end{pmatrix} r\mathrm{d}r\mathrm{d}\theta = \begin{pmatrix} 0 \\ 0 \\ bL\pi R^2 \end{pmatrix}.$$

$$\boldsymbol{T}_{\text{side}} = \iint \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So the total load on all the faces is

$$\boldsymbol{T}_{\text{total}} = \boldsymbol{T}_{\text{top}} + \boldsymbol{T}_{\text{bottom}} + \boldsymbol{T}_{\text{side}} = \begin{pmatrix} 0 \\ 0 \\ bL\pi R^2 \end{pmatrix}.$$

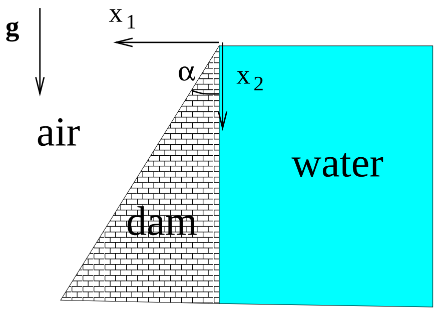
If the body is in equilibrium this must balance the resultant body force:

$$\boldsymbol{B} = \iiint \begin{pmatrix} 0 \\ 0 \\ -b \end{pmatrix} \mathrm{d}V == \begin{pmatrix} 0 \\ 0 \\ -b \end{pmatrix} \iiint \mathrm{d}V = \begin{pmatrix} 0 \\ 0 \\ -b\pi R^2 L \end{pmatrix}.$$

and, as required,

$$\boldsymbol{T}_{\text{total}} + \boldsymbol{B} = \boldsymbol{0}.$$

We note that the body force acts downwards and the resultant stress acts upwards at the base of the cylinder.



2.) We label the vertical face 1 and the sloping face 2, the outer unit normals to these two faces are

$$\boldsymbol{n}^{(1)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{n}^{(2)} = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}.$$

The boundary conditions are $\tau_{ij} n_j = t_i$ on each face.

Face 1: $x_1 = 0$**.**

The traction is given by the hydrostatic pressure acting compressively on the face

$$\boldsymbol{t} = -p\boldsymbol{n}^{(1)} = \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} \rho_w g x_2 \\ 0 \end{pmatrix}.$$

Thus,

$$t_1 = \rho_w g x_2 = \tau_{11} n_1^{(1)} + \tau_{12} n_2^{(1)} = -\tau_{11}.$$

We are given that $\tau_{11} = a x_1 + b x_2$ and so

$$\tau_{11} = a x_1 + b x_2 = 0 + b x_2 = -\rho_w g x_2 \quad \Rightarrow \boxed{b = -\rho_w g.}$$

$$t_2 = 0 = \tau_{21} n_1^{(1)} + \tau_{22} n_2^{(1)} = -\tau_{21}.$$

We are given that $\tau_{21} = \tau_{12} = -d x_1 - a x_2$ and so

$$\tau_{21} = -d x_1 - a x_2 = 0 - a x_2 = 0 \quad \Rightarrow \boxed{a = 0.}$$

Face 2: $x_1 = x_2 \tan\alpha$

There are no forces acting on this face, so the traction is zero. Thus,

$$t_1 = 0 = \tau_{11} n_1^{(2)} + \tau_{12} n_2^{(2)} = \tau_{11}\cos\alpha - \tau_{12}\sin\alpha.$$

We know that $\tau_{11} = -\rho_w g x_2$ and that $\tau_{12} = -d x_1$, so

$$-\rho_w g x_2 \cos\alpha + d x_1 \sin\alpha \quad \Rightarrow \quad \rho_w g x_2 \cos\alpha = d x_2 \tan\alpha \sin\alpha \quad \Rightarrow \quad \boxed{d = \rho_w g \cot^2\alpha}$$

$$t_2 = 0 = \tau_{21} n_1^{(2)} + \tau_{22} n_2^{(2)} = \tau_{21}\cos\alpha - \tau_{22}\sin\alpha.$$

We know that $\tau_{21} = \tau_{12} = -\rho_w g \cot^2\alpha x_1$ and $\tau_{22} = c x_1 + d x_2 - \rho_0 g x_2$, so

$$-\rho_w g \cot^2\alpha\, x_1 \cos\alpha - \left(c x_1 + \rho_w g \cot^2\alpha\, x_2 - \rho_0 g x_2\right)\sin\alpha = 0.$$

$$\Rightarrow \quad -2\rho_w g \cot^2\alpha\, x_2 \sin\alpha - c x_2 \tan\alpha \sin\alpha + \rho_0 g x_2 \sin\alpha = 0.$$

$$\Rightarrow \quad -2\rho_w g \cot^2\alpha - c\tan\alpha + \rho_0 g = 0 \Rightarrow \quad \boxed{c = \cot\alpha\left(\rho_0 g - 2\rho_w g \cot^2\alpha\right)}$$