MATH35021: SOLUTION SHEET III[1]

1.) i.) The two displacement fields $u_i^{(1)}$ and $u_i^{(2)}$ corrrespond to the same strain field e_{ij}. If we let $u_i^{\Delta} = u_i^{(1)} - u_i^{(2)}$ then the corresponding strain field is

$$e_{ij}^{\Delta} = \frac{1}{2}\left(u_{i,j}^{\Delta} + u_{j,i}^{\Delta}\right) = \frac{1}{2}\left(u_{i,j}^{(1)} - u_{i,j}^{(2)} + u_{j,i}^{(1)} - u_{j,i}^{(2)}\right)$$

$$= \frac{1}{2}\left(u_{i,j}^{(1)} + u_{j,i}^{(1)}\right) - \frac{1}{2}\left(u_{i,j}^{(2)} + u_{j,i}^{(2)}\right) = e_{ij} - e_{ij} = 0.$$

ii.) We next use the result from example sheet II that

$$\frac{\partial \omega_{ik}}{\partial x_j} = \frac{\partial e_{ij}}{\partial x_k} - \frac{\partial e_{kj}}{\partial x_i}. \tag{1}$$

If $e_{ij} = 0$, then equation (1) implies that $\omega_{ik,j} = 0$ and hence all componenets ω_{ik} are constants. From the definition,

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \omega_{ij} = \text{constant},$$

and integrating this expression yields

$$u_i = \omega_{ij} x_j + C_i, \tag{2}$$

because ω_{ij} is a constant and C_i is a new constant of integration that corresponds to a rigid-body displacement. Interpreting equation (2), we see that the displacement for a strain of zero consists of a rigid-body rotation plus a rigid-body translation. Hence, any two displacement fields that have the same e_{ij} differ by (at most) a rigid-body motion.

2.) Written in matrix form the strain tensor is

$$e_{ij} = \epsilon(a + b x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We first note that because the strain components depend linearly on x_3 the strain compatibility equations are trivially satisfied because every term therein is a second derivative of the strain. Now, the six partial differential equations for the displacement components are

$$\frac{\partial u_{(i)}}{\partial x_{(i)}} = \epsilon(a + b x_3), \qquad \text{[no summation]},$$

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0, \quad i \neq j.$$

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Integrating the diagonal terms gives

$$u_1 = \epsilon(ax_1 + bx_1x_3) + f_1(x_2, x_3), \quad (3)$$
$$u_2 = \epsilon(ax_2 + bx_2x_3) + f_2(x_1, x_3), \quad (4)$$
$$u_3 = \epsilon(ax_3 + \frac{1}{2}bx_3^2) + f_3(x_1, x_2). \quad (5)$$

The off-diagonal terms now give three relationships between the unknown functions f_i:

$$e_{12} : u_{1,2} + u_{2,1} = 0 \;\Rightarrow\; \frac{\partial f_1}{\partial x_2}(x_2, x_3) + \frac{\partial f_2}{\partial x_1}(x_1, x_3) = 0, \quad (6)$$
$$e_{23} : u_{2,3} + u_{3,2} = 0 \;\Rightarrow\; \epsilon bx_2 + \frac{\partial f_2}{\partial x_3}(x_1, x_3) + \frac{\partial f_3}{\partial x_2}(x_1, x_2) = 0, \quad (7)$$
$$e_{13} : u_{1,3} + u_{3,1} = 0 \;\Rightarrow\; \epsilon bx_1 + \frac{\partial f_1}{\partial x_3}(x_2, x_3) + \frac{\partial f_3}{\partial x_1}(x_1, x_2) = 0. \quad (8)$$

We are not asked for the most general solution only **a** solution and we note that we *can* still satisfy all three equations (6-8) by choosing $f_1 = f_2 = 0$. Then equation (6) is satisfied and equations (7) and (8) become

$$\frac{\partial f_3}{\partial x_2}(x_1, x_2) = -\epsilon bx_2 \quad \text{and} \quad \frac{\partial f_3}{\partial x_1}(x_1, x_2) = -\epsilon bx_1.$$

Integrating these gives

$$f_3(x_1, x_2) = -\frac{1}{2}\epsilon bx_2^2 + g_1(x_1) \quad \text{and} \quad f_3(x_1, x_2) = -\frac{1}{2}\epsilon bx_1^2 + g_2(x_2).$$

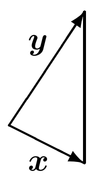
A suitable solution is

$$f_3(x_1, x_2) = -\frac{1}{2}\epsilon b\left(x_1^2 + x_2^2\right).$$

Hence a possible displacement field is

$$u_1 = \epsilon(ax_1 + bx_1x_3),$$
$$u_2 = \epsilon(ax_2 + bx_2x_3),$$
$$u_3 = \epsilon\left(ax_3 + \frac{1}{2}b\left[x_3^2 - x_1^2 - x_2^2\right]\right)$$

3.) The important result that you need to know (or have looked up) is that for a triangle defined by two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, then its area A is given by



$$A = \tfrac{1}{2}|\boldsymbol{x} \times \boldsymbol{y}|.$$

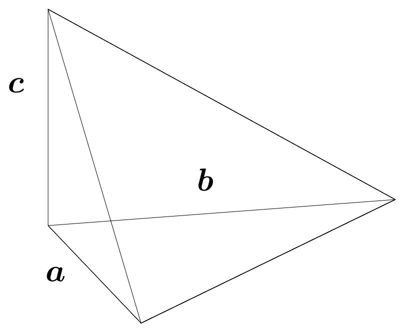


Figure 1: A tetrahedron spanned by three orthogonal vectors. The area of the face spanned by $\boldsymbol{a}$ and $\boldsymbol{c}$ is S_1; the area of the face spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ is S_2; the area of the face spanned by $\boldsymbol{b}$ and $\boldsymbol{c}$ is S_3; and the area of the sloping face is S.

Note that the cross product of the two vectors will be perpendicular to both vectors and so will be normal to the triangle with direction given by the right-hand rule.

The tetrahedron, see Figure 1, can be described by three orthogonal vectors, say $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. The vectors that span the sloping face are $(\boldsymbol{a}-\boldsymbol{c})$ and $(\boldsymbol{b}-\boldsymbol{c})$. Then

$$S\boldsymbol{n} = \frac{1}{2}\left[(\boldsymbol{a}-\boldsymbol{c})\times(\boldsymbol{b}-\boldsymbol{c})\right],$$

where the order of terms in the cross product is chosen to ensure that $\boldsymbol{n}$ is the outer unit normal. Hence,

$$S\boldsymbol{n} = \frac{1}{2}\left(\boldsymbol{a}\times\boldsymbol{b}-\boldsymbol{c}\times\boldsymbol{b}-\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{c}\right).$$

Now $\boldsymbol{c}\times\boldsymbol{c}=0$ and rearranging the remaining terms gives

$$S\boldsymbol{n}+\frac{1}{2}\boldsymbol{b}\times\boldsymbol{a}+\frac{1}{2}\boldsymbol{c}\times\boldsymbol{b}+\frac{1}{2}\boldsymbol{a}\times\boldsymbol{c}=0.$$

Using the above result about areas of triangles and the right-hand rule, we obtain the result

$$S\boldsymbol{n}+S_2\boldsymbol{n_2}+S_3\boldsymbol{n_3}+S_1\boldsymbol{n_1}=0,$$

as required.