MATH35021: SOLUTION SHEET I[1]

1.) Which one of these equations in index notation are valid? Remember the summation convention!

a) $c = a_i b_i$ (OK, this is the dot product $c = \mathbf{a} \cdot \mathbf{b}$)

b) $c = a_{ij} b_i$ (Wrong, the free index j doesn't appear on LHS)

c) $c_i = a_{ij} b_i$ (Wrong, the indices on LHS and RHS don't match)

d) $c_i = a_{ij} b_j$ (OK, this is the matrix vector product with the matrix $\underline{\mathbf{a}}$: $\mathbf{c} = \underline{\mathbf{a}}\mathbf{b}$)

e) $c_i = a_{ji} b_j$ (OK, this is the matrix vector product with the transposed matrix $\underline{\mathbf{a}}$: $\mathbf{c} = \underline{\mathbf{a}}^T\mathbf{b}$)

f) $\sigma_{ij} = \alpha_{ij} T + E_{ijkl} e_{kl}$ (Correct – meet your first 4th order tensor. By the way: this is the constitutive equation for a linearly elastic solid incl. temperature variations)

g) $\sigma_{ij} = \alpha_{kl} T_i + E_{ijkl} e_{ij}$ (Wrong, the indices of all terms are different)

h) $k_{ijkl} = a_i b_{kl} c_{njm} d_{mn} + e_{ik} e_{jn} f_{nl}$ (Messy, but correct)

2.) Using a comma to denote partial differentiation (e.g. $\partial u/\partial x_2 = u_{,2}$), transform the following expressions into index notation:

a) $\nabla u(x_1, x_2, x_3) \to u_{,i}$

b) $\underline{\mathbf{A}} = \nabla \mathbf{u}(x_1, x_2, x_3) \to a_{ij} = u_{i,j}$

c) $\nabla \cdot \mathbf{u}(x_1, x_2, x_3) = f(x_1, x_2, x_3) \to u_{i,i} = f$

d) $\nabla^2 u(x_1, x_2, x_3) = f(x_1, x_2, x_3) \to u_{,ii} = f$

e) $\nabla^2 \mathbf{u}(x_1, x_2, x_3) = \mathbf{f}(x_1, x_2, x_3) \to u_{i,jj} = f_i$

3.) A 2D body occupying the region $\{d : 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$ is displaced by the following displacement fields:

a) $u_1 = \epsilon(x_1 + 2x_2)$; $u_2 = \epsilon(3 + x_2)$ where $\epsilon \ll 1$

b) $u_1 = U_1 - x_1 + \sqrt{x_1^2 + x_2^2}\, \cos\left(\arctan\left(\frac{x_2}{x_1}\right) + \Phi\right)$;
$u_2 = U_2 - x_2 + \sqrt{x_1^2 + x_2^2}\, \sin\left(\arctan\left(\frac{x_2}{x_1}\right) + \Phi\right)$ where U_1, U_2, Φ are constants.

Sketch the deformed body D (setting $\epsilon = 1$ in (a) for simplicity). For the displacement field (a) determine the deformation gradient tensor and the strain and rotation tensors. For the displacement field (b) determine the linear strain and rotation tensors for $\Phi \ll 1$ – Think before you calculate! [Hints: Does anything here smell of cylindrical polars? First consider the two special cases $U_1 = U_2 = 0$ and $\Phi = 0$.]

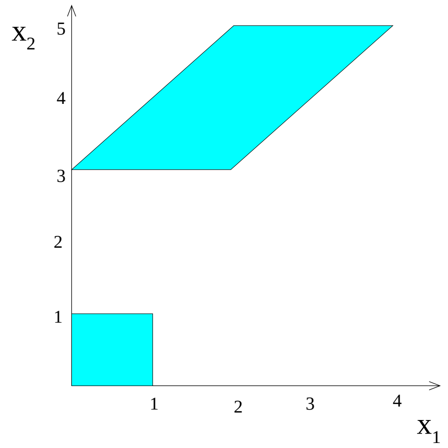
[1] Any feedback to: *Andrew.Hazel@manchester.ac.uk*

Solution:

a) For $\epsilon = 1$:

$$\mathbf{R} = \mathbf{r} + \mathbf{u} = (2(x_1 + x_2),\ 3 + 2x_2)^T$$

Use this to sketch the edges (for $\epsilon = 1$, say), e.g. $\mathbf{R}(0,0) = (0,3)^T$, $\mathbf{R}(1,1) = (4,5)^T$, etc.



For arbitrary (small) ϵ:

$$u_{i,j} = \epsilon \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad e_{ij} = \epsilon \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \omega_{ij} = \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

b) The body is displaced by $(U_1, U_2)^T$ and then rotated by Φ (about the $\mathbf{e}_3$ axis). Therefore for small rotations and a linearised theory:

$$e_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \omega_{ij} = \begin{pmatrix} 0 & -\Phi \\ \Phi & 0 \end{pmatrix}$$

This can be confirmed by linearisation of e_{ij} and ω_{ij} w.r.t. Φ. The easiest way to perform this calculation is to work in a mixture of Cartesian and plane polar coordinates (not to be recommended unless you know **exactly** what you are doing!). Converting to plane polars we have that

$$r = \sqrt{x_1^2 + x_2^2}, \quad \text{and} \quad \theta = \arctan\left(\frac{x_2}{x_1}\right),$$

and so

$$u_1 = U_1 - x_1 + r\cos(\theta + \Phi),$$
$$u_2 = U_2 - x_2 + r\sin(\theta + \Phi).$$

We now use the appropriate trigonometric identities to write

$$u_1 = U_1 - x_1 + r\left(\cos\theta\cos\Phi - \sin\theta\sin\Phi\right),$$

$$u_2 = U_2 - x_2 + r\left(\sin\theta\cos\Phi + \cos\theta\sin\Phi\right);$$

and convert back to Cartesians $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ (sneaky isn't it)

$$u_1 = U_1 - x_1 + x_1\cos\Phi - x_2\sin\Phi,$$

$$u_2 = U_2 - x_2 + x_2\cos\Phi + x_1\sin\Phi.$$

In this form, the differentiation is straightforward

$$\frac{\partial u_1}{\partial x_1} = -1 + \cos\Phi, \quad \frac{\partial u_1}{\partial x_2} = -\sin\Phi,$$

$$\frac{\partial u_2}{\partial x_1} = \sin\Phi, \quad \frac{\partial u_2}{\partial x_2} = -1 + \cos\Phi,$$

and so

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} \end{pmatrix}$$
$$= \begin{pmatrix} -1+\cos\Phi & 0 \\ 0 & -1+\cos\Phi \end{pmatrix};$$

and

$$\omega_{ij} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin\Phi \\ \sin\Phi & 0 \end{pmatrix}.$$

Finally, we make the approximation that $|\Phi| \ll 1$, so that $\sin\Phi \approx \Phi$ and $\cos\Phi \approx 1$, to obtain, as promised,

$$e_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \omega_{ij} = \begin{pmatrix} 0 & -\Phi \\ \Phi & 0 \end{pmatrix}$$

Sketch for $\Phi = \pi/6, U_1 = 3, U_2 = 2$: