Computation of the z-radical in $C(X)$

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Abstract. We say that a Tychonoff space $X$ has computable $z$-radicals if for all ideals $a$ of $C(X)$, the smallest $z$-ideal containing $a$ is generated as an ideal by all the $s \circ f$, where $f$ is in $a$ and $s$ is a continuous function $\mathbb{R} \to \mathbb{R}$ with $s^{-1}(0) = \{0\}$. We show that every cozero set of a compact space has computable $z$-radicals and that a subset $X$ of $\mathbb{R}^n$ has computable $z$-radicals if and only if $X$ is locally closed.

1. Introduction

An ideal $a$ of the ring $C(X)$ of continuous functions from a Tychonoff space $X$ (i.e., $X$ is completely regular) into $\mathbb{R}$ is a $z$-ideal if $f \in a$ whenever $f$ vanishes on a zero set of a function from $a$. The $z$-radical $\sqrt{a}$ of an arbitrary ideal $a$ of $C(X)$ is the smallest $z$-ideal of $C(X)$ containing $a$.

In this paper we want to compute $\sqrt{a}$ from $a$. This needs some explanation. First consider the ordinary radical $\sqrt{a}$ of $a$. We have $\sqrt{a} = \{ f \in C(X) \mid f^n \in a \text{ for some } n \in \mathbb{N} \}$. Hence membership of a given $f \in C(X)$ in $\sqrt{a}$ can be tested by applying certain continuous functions $s : \mathbb{R} \to \mathbb{R}$ with $s^{-1}(0) = \{0\}$ to $f$ - namely the functions $s(x) = x^n$, and then to check if $s \circ f$ is in $a$. Now if $s : \mathbb{R} \to \mathbb{R}$ is any continuous function with $s^{-1}(0) = \{0\}$ and $s \circ f \in a$, then $f \in \sqrt{a}$, since $f$ and $s \circ f$ have the same zeros. The problem in question is, if this is the appropriate test for membership in $\sqrt{a}$.

Let $\Upsilon := \{ s : \mathbb{R} \to \mathbb{R} \mid s \text{ is continuous and } s^{-1}(0) = \{0\} \}$ (the Greek letter “Upsilon”) and suppose that for a given ideal $a$ we have $\sqrt{a} = \{ f \in C(X) \mid s \circ f \in a \text{ for some } s \in \Upsilon \}$. Then it turns out (cf. (5.13)) that $\sqrt{a} = \{ g \circ (s \circ f) \mid f \in a, \ g \in C(X), \ s \in \Upsilon \}$ - and in this sense we consider $\sqrt{a}$ as “computable” from $a$. Note that the ordinary radical is computed quite similarly: $\sqrt{a} = \{ g \cdot \sqrt{|f|} \mid f \in a, \ g \in C(X), \ n \in \mathbb{N} \}$. So we may view $\Upsilon$ as a set, generalizing the power function $x^n$ and the root functions $\sqrt{|x|}$ at the same time.

We say that a Tychonoff space $X$ has computable $z$-radicals if $\sqrt{a} = \{ f \in C(X) \mid s \circ f \in a \text{ for some } s \in \Upsilon \}$ for all ideals $a$ of $C(X)$. By (8.10) and (8.18) below, we have:

(1.1) Theorem. If $X$ is a cozero set of a compact space, then $X$ has computable $z$-radicals.

and

(1.2) Theorem. If $X \subseteq \mathbb{R}^n$, then $X$ has computable $z$-radicals if and only if $X$ is locally closed.

The proofs of both theorems rely on two other constructions attached to ideals $a$ of $C(X)$. The first one is the construction $\sqrt[\Upsilon]{a} = \{ f \in C(X) \mid s \circ f \in a \text{ for some } s \in \Upsilon \}$ for an ideal $a$ of $C(X)$: we prove that $\sqrt[\Upsilon]{a}$ is the smallest ideal containing $a$, which is closed under

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composition with functions from $\mathcal{Y}$ (cf. (5.8) and (5.13)(ii)). This allows to reduce the question whether $X$ has computable $z$-radicals to the question if all prime ideals $\mathfrak{p}$ of $C(X)$, which are $\mathcal{Y}$-radical (i.e. $\sqrt[\mathcal{Y}]{\mathfrak{p}} = \mathfrak{p}$) are $z$-radical.

The second construction concerns the formation of the inverse to the various radical notions above. Let $R$ be one of the symbols $\sqrt{\_}$, $\sqrt[\mathcal{Y}]{\_}$, $\sqrt[\mathcal{Z}]{\_}$. Then for each ideal $\mathfrak{a}$ of $C(X)$, there is a largest $R$-radical ideal contained in $\mathfrak{a}$: for the ordinary radical this is an immediate consequence of RCR 2 in section 2 below, for the other radical notions this is (3.8) and (5.7) below. In fact it is easier to compute first these inverse radicals associated to an ideal, than to compute the corresponding radical directly (cf. (5.7) and (9.2)).

Geometrically, a Tychonoff space $X$ has computable $z$-radicals if and only if the Nullstellensatz holds for all principal ideals $(f)$ of $C(X)$, in the sense that $\{f = 0\} \subseteq \{g = 0\}$ if and only if there is some $s \in \mathcal{Y}$ with $s \circ g \in (f)$.

In order to obtain a geometrical meaning for the computation of the inverse radicals we analyze in section 4 another construction attached to ideals of $C(X)$. This is the so-called tubular ideal $O(\mathfrak{a})$ associated to an ideal $\mathfrak{a}$ of $C(X)$, which defines a canonical infinitesimal tubular neighborhood of $V(\mathfrak{a})$ in Spec $C(X)$. If $\mathfrak{a}$ is a maximal ideal, then $O(\mathfrak{a})$ corresponds to the Gillman-Jerison "$O^\mathcal{P}$". If $\mathfrak{a} = (f)$ is a principal ideal and $X$ is a normal space, then $g \in O(\mathfrak{a})$ if and only if the zero set of $f$ is contained in the interior of the zero set of $g$.

In (9.2) we show that the inverse $\mathcal{Y}$-radical and the inverse $z$-radical of a finitely generated ideal $\mathfrak{a}$ of $C(X)$ agree if $X$ satisfies a mild condition, in particular if $X$ is locally compact or a metric space. If in addition $X$ is normal, then both ideals are equal to the tubular ideal $O(\mathfrak{a})$.

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2. Spectral spaces and real algebra

In this section we recall basics about spectral spaces and real algebra. The references are [Gr] (where spectral spaces are called "Stone-spaces") and the books [BCR] and [Kn-Sch], where most of the material about spectral spaces of this section can be found. At the end of this section we recall some facts from the paper [Schw2] about the use of these tools for rings of continuous functions.

(2.1) Notation. Let $X$ be any topological space. If $x, y \in X$ we write $x \rightsquigarrow y$ if $y \in \overline{x}$ and we say $y$ is a specialization of $x$ or $x$ is a generalization of $y$. We write $X^{\min}$ for the elements of $X$, which are minimal w.r.t. $\rightsquigarrow$ and $X^{\max}$ for the elements of $X$, which are maximal w.r.t. $\rightsquigarrow$.

If $Y \subseteq X$, then $\text{int}_X Y$ denotes the interior of $Y$ in $X$ and we suppress the subscript $X$ if the ambient space is clear from the context.

We define $\mathcal{K}(X) := \{U \subseteq X \mid U \text{ open and quasi-compact}\}$, $\overline{\mathcal{K}}(X) := \{X \setminus U \mid U \in \mathcal{K}(X)\}$ and $\mathcal{K}(X) := \text{the boolean algebra of subsets of } X$, generated by $\mathcal{K}(X) \cup \overline{\mathcal{K}}(X)$.

(2.2) Definition. (cf. [Ho]) A spectral space is a topological space $X$, which is quasi-compact and $T_0$, such that $\mathcal{K}(X)$ is a basis of $X$, closed under finite intersections and such that every closed and irreducible subset $A \subseteq X$ has a generic point, i.e. $A = \{x\}$ for some $x \in A$. 

The prime spectrum $\text{Spec } A$ of a unital, commutative ring $A$ is a spectral space. Hochster’s Theorem (cf. [Ho]) says that every spectral space is homeomorphic (but not canonically homeomorphic!) to the spectrum of a unital, commutative ring.

(2.3) Definition. If $X$ is a spectral space, then another topology is defined on $X$, which has $\mathcal{K}(X) \cup \overline{\mathcal{K}}(X)$ as a subbasis of open sets. This topology is called the constructible topology and $X^{\text{con}}$ denotes $X$ when viewed with this topology. A subset of $X$ which is closed and open in $X^{\text{con}}$ is called constructible. The closed subsets of $X^{\text{con}}$ are called proconstructible.

The first fundamental theorem on spectral spaces says that $X^{\text{con}}$ is a boolean space. Hence $X^{\text{con}}$ is quasi–compact, Hausdorff and totally disconnected. It follows that the constructible subsets of $X$ are exactly those from $\mathcal{K}(X)$ and that $\overline{\mathcal{K}}(X)$ is precisely the set of all closed, constructible subsets of $X$.

(2.4) Definition. If $X$ is a spectral space then the inverse topology on $X$ is defined as the topology on $X$, which has $\overline{\mathcal{K}}(X)$ as a basis of open sets. This space is denoted by $X^\text{opp}$.

$X^{\text{opp}}$ is again spectral, $\mathcal{K}(X^{\text{opp}}) = \overline{\mathcal{K}}(X)$, $\mathcal{K}(X^{\text{opp}}) = \mathcal{K}(X)$ and $\mathcal{K}(X^{\text{opp}}) = \mathcal{K}(X)$. If $x, y \in X$, then $x \sim y$ if and only if $y \sim x$ in $X^{\text{opp}}$.

(2.5) Definition. A map $f : X \rightarrow Y$ between spectral spaces $X$ and $Y$ is called spectral if $f^{-1}(V) \in \mathcal{K}(X)$ for all $V \in \mathcal{K}(Y)$. In other words, $f$ is spectral iff $f$ is continuous and with respect to the constructible topologies.

An important fact is that the proconstructible subsets of a spectral space $X$ are precisely the subsets $Y$ of $X$ which are spectral in the induced topology and for which the inclusion $Y \rightarrow X$ is a spectral map.

The category of spectral spaces has the spectral maps as morphisms. We now describe the Stone duality for spectral spaces. Let $L = (L, \wedge, \vee, 0, 1)$ be a distributive lattice with smallest element $0$ and largest element $1$. Here, we always assume that lattices contain a least element $0$, a largest element $1$ and that all lattice homomorphisms map $0$ to $0$ and $1$ to $1$. Let $\text{Prim } L$ be the set of prime filters $\mathcal{f}$ of $L$ (a proper filter $\mathcal{f}$ is prime if it contains $a$ or $b$ whenever $a \vee b \in \mathcal{f}$). We view $\text{Prim } L$ as a topological space where a subbasis of open sets consists of all $D(a) := \{ \mathcal{f} \in \text{Prim } L \mid a \notin \mathcal{f} \}$. It turns out that $\text{Prim } L$ is a spectral space with $\mathcal{K}(\text{Prim } L) := \{ D(a) \mid a \in L \}$. We write $V(a) := \text{Prim } L \setminus D(a)$, hence $\overline{\mathcal{K}}(\text{Prim } L) = \{ V(a) \mid a \in L \}$ is a lattice of subsets of $\text{Prim } L$. The Stone representation says that the map $L \rightarrow \overline{\mathcal{K}}(\text{Prim } L)$, which sends $a$ to $V(a)$ is a lattice isomorphism (respecting $0$ and $1$).

Now we describe the anti–equivalence between spectral spaces and distributive lattices (with $0$ and $1$). For a lattice homomorphism $\varphi : L \rightarrow L'$ the map $\text{Prim } \varphi : \text{Prim } L' \rightarrow \text{Prim } L$, $\text{Prim } \varphi(\mathcal{f}') := \varphi^{-1}(\mathcal{f}')$ is a spectral map and $\text{Prim }$ is a contravariant functor from the category of distributive lattices into the category of spectral spaces.

The second fundamental theorem on spectral spaces says that $\text{Prim } L$ is an anti-equivalence; the inverse is given by $X \mapsto \overline{\mathcal{K}}(X)$ for a spectral space $X$.

I want to add a third fundamental property, which is a separation property that is responsible for many arguments related to spectral spaces. I did not find a reference for it, so the proof is included:
(2.6) Theorem. (Separation Lemma)
Let $X$ be a spectral space and let $Y, Z \subseteq X$ be such that $Z$ is quasi-compact and $Y$ is quasi-compact in the inverse topology. Then the following are equivalent.

(i) for all $y \in Y$ and all $z \in Z$, $y \not\rightarrow z$.

(ii) there is a closed, constructible set $A \subseteq X$ with $Y \subseteq A$ and $A \cap Z = \emptyset$.

Proof. Clearly (ii) implies (i).

(i) $\Rightarrow$ (ii). First we claim that for each $y \in Y$ there is a closed and constructible set $A_y \subseteq X$ with $y \in A_y$ and $A_y \cap Z = \emptyset$. To see this, we pick some $y \in Y$. By (i), for every $z \in Z$ there is a closed and constructible set $B_z \subseteq X$ with $y \in B_z$ and $z \not\in B_z$. Hence $Z \subseteq \bigcup_{z \in Z} X \setminus B_z$. Since $Z$ is quasi-compact and the $X \setminus B_z$ are open, there are $z_1, \ldots, z_k \in Z$ with $Z \subseteq (X \setminus B_{z_1}) \cup \ldots \cup (X \setminus B_{z_k})$. Hence $A_y := B_{z_1} \cap \ldots \cap B_{z_k}$ is disjoint from $Z$ and $A_y$ is a closed and constructible set containing $y$. This shows our claim.

Now we find $A$ as follows. We have $Y \subseteq \bigcup_{y \in Y} A_y$. Since $Y$ is quasi-compact in the inverse topology and the $A_y$ are closed and constructible, there are $y_1, \ldots, y_n \in Y$ with $Y \subseteq A := A_{y_1} \cup \ldots \cup A_{y_n}$. Clearly $A$ is disjoint from $Z$, closed and constructible. \qed

Recall that a proconstructible subset $Y$ of a spectral space $X$ is quasi-compact and quasi-compact in the inverse topology. The converse is highly false, e.g. every subset $Y$ of $X$ containing $X^{\text{max}} \cup X^{\text{min}}$ has these properties (observe that any subset of $X$ containing $X^{\text{max}}$ is quasi-compact, since every point of $X$ specializes to a point in $X^{\text{max}}$), but most of them are not proconstructible.

Frequently used properties of spectral spaces follow quickly from the Separation Lemma.

(2.7) Corollary. Let $X$ be a spectral space and let $Y, Z \subseteq X$. Then

(i) If $Y$ is quasi-compact in the inverse topology, then $\overline{Y} = \bigcup_{y \in Y} \{y\}$.

(ii) If $Y$ is closed, $Z$ is quasi-compact and disjoint from $Y$, then there is a closed, constructible subset $A$ of $X$ with $Y \subseteq A$ and $A \cap Z = \emptyset$.

(iii) If $Y$ and $Z$ are quasi-compact in the inverse topology and if there are no points $y \in Y$, $z \in Z$ which have a common specialization in $X$, then there are closed and constructible subsets $A, B$ of $X$ with $Y \subseteq A$, $Z \subseteq B$ and $A \cap B = \emptyset$.

(iv) If $Y$ and $Z$ are quasi-compact and if there are no points $y \in Y$, $z \in Z$ which have a common generalization in $X$, then there are open quasi-compact subsets $U, V$ of $X$ with $Y \subseteq U$, $Z \subseteq V$ and $U \cap V = \emptyset$.

Proof. (i). If $z \in \overline{Y}$, then by (2.6) applied to $Z = \{z\}$ there must be some $y \in Y$ with $y \rightarrow z$. Hence $\overline{Y} = \bigcup_{y \in Y} \{y\}$. Item (ii) follows immediately from (2.6).

(iii). By (i), $\overline{Y} = \bigcup_{y \in Y} \{y\}$ and $\overline{Z} = \bigcup_{z \in Z} \{z\}$. Hence by assumption, $\overline{Y} \cap \overline{Z} = \emptyset$ and by (ii) there is a closed, constructible set $A \subseteq X$ with $Y \subseteq A$ and $A \cap \overline{Z} = \emptyset$. By (ii) applied to $\overline{Z}$ and $A$ there is a set $B$ as required.

(iv) is (iii) for the inverse spectral space of $X$. \qed

A direct consequence of (2.7)(iv) is that $X^{\text{min}}$ is Hausdorff.

If $X = \text{Spec} A$ and $S \subseteq A$, then we use the standard notation $V(S) := \{p \in \text{Spec} A \mid S \subseteq p\}$ and $D(S) := \{p \in \text{Spec} A \mid S \cap p = 0\}$.

This will not interfere with the same notations in Prim $L$ for a distributive lattice $L$ - on the contrary: the lattice notations are inspired by the commutative algebra setup.

For our purposes a certain subcategory of spectral spaces plays a major role:
Recall that a topological space $X$ is called normal if for all disjoint closed subsets $Y, Z$ of $X$, there are disjoint open subsets $U, V$ of $X$ with $Y \subseteq U$ and $Z \subseteq V$. If $X$ is a spectral space, then $X$ is normal if and only if for all $x \in X$ there is a unique $y \in X^{\text{max}}$ such that $x \rightsquigarrow y$. In this case $X^{\text{max}}$ is a Hausdorff space. All this is well known and follows quickly from (2.7).

Moreover if $X$ is a normal spectral space, then the map $r : X \longrightarrow X^{\text{max}}$ which sends $x$ to the unique maximal specialization of $x$ in $X$ is continuous and closed, so $r$ is a retract of the inclusion $X^{\text{max}} \longrightarrow X$.

(2.8) Definition. A spectral space $X$ is called completely normal if for all $x, y, z \in X$ with $x \rightsquigarrow y, z$ we have $y \rightsquigarrow z$ or $z \rightsquigarrow y$.

By (2.7)(iv) again, a spectral space $X$ is completely normal, if and only if every proconstructible subset of $X$ is normal.

The typical example of a completely normal spectral space is the real spectrum $\text{Sper} A$ of a ring $A$, but in contrast to the Hochster result, not every completely normal spectral space is the real spectrum of a ring, cf. [Del-Ma]. The real spectrum only pops up implicitly in the current paper, since we are mainly concerned with rings that have a real spectrum, canonically homeomorphic to $\text{Spec} A$. This is explained next.

Most of the algebraic constructions from commutative ring theory cannot be performed inside the category of rings of continuous functions. A category that has this flexibility and which is still close to rings of continuous functions is the category of real closed rings. These rings have been discovered by Niels Schwartz (cf. [Schw1]) in the context of real algebraic geometry. A ring (in this paper we always mean unital and commutative ring) is called real closed if $A$ is a reduced $f$-ring such that the squares of $A$ are the set of all $f \in A$ with $f \geq 0$, such that $g|f^2$, whenever $0 \leq f \leq g$ and such that for each prime ideal of $A$, the ring $A/p$ is integrally closed with real closed quotient field. One may think that a real closed ring replaces the notion of a real closed field if we switch from ordered fields to reduced $f$-rings; the situation is much more involved though. The book [Schw-Ma] is a rich source of categorical constructions that are possible with real closed rings.

There are many other descriptions of real closed rings. For us it is enough to know the following facts about a real closed ring $A$ (cf. [Schw1], [Schw2] and [Schw-Ma]):

RCR 1. The support map $\text{supp} : \text{Sper} A \longrightarrow \text{Spec} A$ is a homeomorphism, in particular, $\text{Spec} A$ is a completely normal spectral space. So for example every radical ideal of $A$ that contains a prime ideal is prime.

RCR 2. If $a, b$ are ideals of $A$, then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, hence for each ideal $a$ of $A$ there is a largest radical ideal contained in $a$.

RCR 3. If $a$ is a radical ideal of $A$, then $a$ is convex and $A/a$ is again real closed.

RCR 4. If $X$ is a Tychonoff space, then $C(X)$ and the ring $C^*(X)$ of bounded continuous functions $X \longrightarrow \mathbb{R}$ are real closed. Moreover if $X$ is a semi-algebraic subset of $\mathbb{R}^n$, then the ring of continuous semi-algebraic maps $X \longrightarrow \mathbb{R}$ is real closed. A semi-algebraic map is a map whose graph is a semi-algebraic set, i.e. it is a boolean combination of sets of the form $\{f \geq 0\}$, where $f$ is a polynomial with coefficients in $\mathbb{R}$.
Facts and Notations in $C(X)$

Let $X$ be a Tychonoff space, i.e. a completely regular Hausdorff space. Let $C(X)$ be the ring of continuous functions $X \rightarrow \mathbb{R}$. Then $C(X)$ is a subring and a sublattice of the distributive lattice $\mathbb{R}^X$. Recall that for every topological space $Y$ there is a Tychonoff space $X$ such that $C(X)$ is isomorphic to $C(Y)$ (cf. [Gil-Jer], 3.9). Moreover Tychonoff spaces are precisely the subspaces of compact spaces.

A zero set of $X$ is a set of the form $\{ f = 0 \} := \{ x \in X \mid f(x) = 0 \}$, with $f \in C(X)$. A cozero set of $X$ is a set of the form $\{ f \neq 0 \} := \{ x \in X \mid f(x) \neq 0 \}$, with $f \in C(X)$. Observe that any set of the form $\{ f \geq 0 \}$ with $f \in C(X)$ is a zero set, since $\{ f \geq 0 \} = \{ f \wedge 0 = 0 \}$.

If $x \in X$, then the set $\hat{x} := \{ f \in C(X) \mid f(x) = 0 \}$ is obviously a maximal ideal of $C(X)$. The map $X \rightarrow \text{Spec } C(X)$ which sends $x$ to $\hat{x}$ is an homeomorphism of $X$ onto the subset $\hat{X} := \{ \hat{x} \mid x \in X \}$ equipped with the induced topology of $\text{Spec } C(X)$; here we need that $X$ is a Tychonoff space, because in a Tychonoff space the zero sets form a basis of closed sets of the topology of $X$. We shall identify $X$ with $\hat{X}$. Spec $C(X)$ is the closure of $X$, since $D(f) \neq \emptyset$ implies $f \neq 0$. Spec $C(X)$ is also the closure of $X$ in the inverse topology, since $V(f) \neq \emptyset$ implies that $f$ is a non unit - hence it must have a zero in $X$.

Let $z\cdot\text{Spec } C(X)$ be the set of all prime $z$-ideals. By definition of “$z$-ideal” we have

$$(*)\quad z\cdot\text{Spec } C(X) = \bigcap_{f,g \in C(X), \{ f=0 \} \subseteq \{ g=0 \}} D(f) \cup V(g),$$

hence $z\cdot\text{Spec } C(X)$ is the closure of $\hat{X}$ in the constructible topology of $\text{Spec } C(X)$ (cf. [Schw2], section 3).

==LONG VERSION==

(If $p \in z\cdot\text{Spec } C(X)$ and $p \in V(f) \cap D(g)$, then by $(*)$ we can not have $\{ f = 0 \} \subseteq \{ g = 0 \}$, so $V(f) \cap D(g)$ contains an element from $\hat{X}$).

==END OF LONG VERSION==

We recall another useful description of $z\cdot\text{Spec } C(X)$ from [Gil-Jer]: The set $L_X$ of all zero sets of $X$ is a sublattice of the power set of $X$ and the map $\text{Prim } L_X \rightarrow \text{Spec } C(X)$ which sends a prime filter $\mathcal{f}$ of $L_X$ to the set $\{ f \in C(X) \mid \{ f = 0 \} \in \mathcal{f} \}$ is an homeomorphism onto $z\cdot\text{Spec } C(X)$. It is easy to see that the maximal and the minimal points of $\text{Spec } C(X)$ are in $z\cdot\text{Spec } C(X)$ (cf. [Schw2], section 3). We define $\beta X := \text{Spec } C(X)^\max$ equipped with the topology induced by $\text{Spec } C(X)$. Recall that $\beta X \supseteq \hat{X} \cong X$ is the Stone-Čech compactification of $X$.

If $f \in C(X)$, then by $(*)$ again, $V(f) \cap z\cdot\text{Spec } C(X)$ is the closure of $\{ f = 0 \}$ in $z\cdot\text{Spec } C(X)$ (observe that $V(f)$ is in general not the closure of $\{ f = 0 \}$ in $\text{Spec } C(X)$). In particular the closure $\{ f = 0 \}^{z\cdot\text{Spec } C(X)}$ of the zero set of $f$ in $\beta X$ is $V(f) \cap \beta X$.

3. The diamond
We start by defining the \( z \)-radical of an ideal \( a \) and its inverse radical, which we call “diamond”.

(3.1) **Proposition.** Let \( f, g, h \in C(X) \) with \( \{ f = 0 \} \cap \{ g = 0 \} \subseteq \{ h = 0 \} \). Then there are \( f_1, g_1 \in C(X) \) with \( \{ f = 0 \} \subseteq \{ f_1 = 0 \} \) and \( \{ g = 0 \} \subseteq \{ g_1 = 0 \} \) such that \( h = f_1 + g_1 \).

**Proof.** By [Ru], Lemma 3.1.

\[ \square \]

(3.2) **Lemma.** Let \( A \) be a ring of functions \( X \rightarrow R \) from a set \( X \) into a real closed field \( R \). The following are equivalent.

(i) If \( z, n, f \in A \) such that for all \( x \in \{ z = 0 \} \cap \{ n = 0 \} \) we have \( f(x) = 0 \), then there is some \( F \in A \) such that \( F = f \) on \( \{ z = 0 \} \) and \( F = 0 \) on \( \{ n = 0 \} \).

(ii) For all \( f, g, h \in A \) with \( \{ f = 0 \} \cap \{ g = 0 \} \subseteq \{ h = 0 \} \) there are \( f_1, g_1 \in A \) with \( h = f_1 + g_1 \) and \( \{ f = 0 \} \subseteq \{ f_1 = 0 \} \), \( \{ g = 0 \} \subseteq \{ g_1 = 0 \} \)

**Proof.** (i)\( \Rightarrow \) (ii). We have \( h(x) = 0 \) for all \( x \in \{ f = 0 \} \cap \{ g = 0 \} \). By (i) there is some \( H \in A \) with \( H = h \) on \( \{ f = 0 \} \) and \( H = 0 \) on \( \{ g = 0 \} \). Then \( f_1 = h - H \) and \( g_1 = H \) have the required properties.

(ii)\( \Rightarrow \) (i). We have \( \{ z = 0 \} \cap \{ n = 0 \} \subseteq \{ f = 0 \} \), hence by (ii) there are \( z_1, n_1 \in A \) such that \( f = z_1 + n_1 \), \( \{ n = 0 \} \subseteq \{ n_1 = 0 \} \) and \( \{ z = 0 \} \subseteq \{ z_1 = 0 \} \). So \( F := n_1 \) has the required properties. \( \square \)

We’ll not use the lemma, as we do not know how.

(3.3) **Notation.** Let \( Y, E \subseteq X \) be subsets of a Tychonoff space \( X \) and let \( f : Y \rightarrow R \) be a map. We say that \( f \) is locally bounded at \( E \) if for all \( x \in E \) there is an open neighborhood \( U \) of \( x \) in \( X \) and some \( c \in R, c > 0 \) such that \( |f(u)| \leq c (u \in U \cap Y) \).

(3.4) **Proposition.** Let \( X \) be a Tychonoff space and let \( X = Y \cup Z \). If \( g \in C(Y), h \in C(Z) \) are locally bounded at \( Y \cap Z, \) then for every \( f \in C(X) \) which vanishes on \( Y \cap Z, \) there is a unique \( \varphi \in C(X) \) with \( \varphi|_Y = f|_Y \cdot g \) and \( \varphi|_Z = f|_Z \cdot h. \) We have \( \{ \varphi = 0 \} = \{ f = 0 \} \cup \{ g = 0 \} \cup \{ h = 0 \} \).

**Proof.** We have to define

\[ \varphi : X \rightarrow R \]

\[ x \mapsto \begin{cases} f(x) \cdot g(x) & \text{if } x \in Y \\ f(x) \cdot h(x) & \text{if } x \in Z \end{cases} \]

Let \( S = Y \cap Z \). Clearly \( \varphi \) is continuous on \( X \setminus S \). For \( s \in S, \lim_{x \rightarrow s} \varphi(x) = 0, \) since \( g \) and \( h \) are locally bounded at \( S \) and \( f \in C(X) \) vanishes on \( S \). Hence \( \varphi \) is continuous. \( \square \)

We denote \( \varphi \) by \( f \odot (g, h). \) If \( f \) is zero on \( Z, \) then we write \( f \odot g \) instead of \( f \odot (g, h). \)

(3.5) **Corollary.** Let \( f, g, h \in C(X) \) with \( E := \{ f = 0 \} \cap \{ g = 0 \} \subseteq \{ h = 0 \} \). Let \( \psi_f \in C(X \setminus E) \) be defined by \( \psi_f(x) = \frac{f(x)^2}{f(x)^2 + g(x)^2} \) and let \( \psi_g \in C(X \setminus E) \) be defined by \( \psi_g(x) = \frac{g(x)^2}{f(x)^2 + g(x)^2} \).

Then

\[ h = h \odot \psi_f + h \odot \psi_g. \]

More intuitively we write

\[ h = h \odot \frac{f^2}{f^2 + g^2} + h \odot \frac{g^2}{f^2 + g^2}. \]
Hence $f_1 := h \odot \frac{f^2}{f + g^2}$ and $g_1 := h \odot \frac{g^2}{f + g^2}$ are continuous functions on $X$ with $h = f_1 + g_1$, $|f_1|, |g_1| \leq |h|$, $\{f = 0\} \subseteq \{f_1 = 0\}$, $\{g = 0\} \subseteq \{g_1 = 0\}$ and $\{h = 0\} = \{f_1 = 0\} \cap \{g_1 = 0\}$.

**Proof.** Since $\psi f$ is bounded on $X \setminus E$ and $h$ vanishes on $E$ we can apply (3.4). So $h \odot \psi f$ is well defined. The same argument shows that $h \odot \psi g$ is well defined; all other statements are obvious.

The corollary above can also be found in [Ru], Lemma 3.1.

\[ \text{Definition.} \quad \text{If } a \text{ is an ideal of } C(X), \text{ then the } \mathbf{z}\text{-radical of } a \text{ (cf. [Gil-Jer], 2.7) is defined as} \]

\[ \sqrt{a} := \{ f \in C(X) \mid \{ f = 0 \} \supseteq \{ g = 0 \} \text{ for some } g \in a \}. \]

Hence $\sqrt{a}$ is the smallest $z$-ideal of $C(X)$ containing $a$. Therefore we call $z$-ideals also $\mathbf{z}$-radical ideals.

**Corollary.** Let $a, b \subseteq C(X)$ be ideals. Then $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$.

**Proof.** This is [Gil-Jer], 14.8. and follows easy from (3.5) (cf. [Ru], Thm 4.1).

\[ \text{Definition.} \quad \text{Corollary (3.7) implies that the sum of two } \mathbf{z}\text{-radical ideals is again } \mathbf{z}\text{-radical. With Zorn, this assertion implies (is even equivalent) that for every ideal } a \text{ of } C(X), \text{ there is a largest } \mathbf{z}\text{-radical ideal } a^\circ \text{ of } C(X), \text{ contained in } a. \text{ We call } a^\circ \text{ the } \mathbf{diamond} \text{ of } a, \text{ since it has brilliant properties. Proposition (3.5) allows a direct description of } a^\circ:\]

\[ a^\circ = \{ f \in C(X) \mid \text{for all } g \in C(X) \text{ with } \{ f = 0 \} \subseteq \{ g = 0 \} \text{ we have } g \in a \}. \]

**Proof.** Let $b$ be the set on the right hand side. By definition, if $f \in b$ and $\{ f = 0 \} \subseteq \{ g = 0 \}$, then $g \in b$. Clearly $\text{C}(X) \subseteq b$ and every $z$-ideal contained in $a$ is contained in $b$, too. It remains to show that $b$ is closed under addition. This holds by (3.5).

**Remark.** If $X$ is a metric space or a locally compact space and $f, g \in C(X)$, then by (9.2) below, $g \in (f)^\circ$ is equivalent to $\{ f = 0 \} \subseteq \text{int}\{ g = 0 \}$.

**Corollary.** Let $a$ be an ideal of $C(X)$. Then

(i) If $a$ is prime, then $a^\circ$ is prime.

(ii) Each $p \in V(\sqrt{a})^{\min}$ is of the form $\sqrt{q}$ for some $q \in V(a)^{\min}$. Moreover $p = q + \sqrt{a}$ for all such prime ideals $q$. In particular, $p$ is $z$-radical.

(iii) Each $p \in V(a)^{\min}$ is of the form $q + \sqrt{a}$ for some $q \in V(a^\circ)^{\min}$. In particular $\sqrt{a} = \bigcap_{q \in V(a^\circ)^{\min}} q + \sqrt{a}$.

(iv) Each $p \in V(\sqrt{a})^{\min}$ is of the form $q + \sqrt{a}$ for some $q \in V(a^\circ)^{\min}$. 

\[ \text{Definition.} \quad \text{Let } a, b \subseteq C(X) \text{ be ideals. Then } \text{compatibility} (a, b) := \{ f \in C(X) \mid f \text{ is } \mathbf{z}\text{-ideal of } a \text{ and } f \text{ is } \mathbf{z}\text{-ideal of } b \}. \]

\[ \text{Definition.} \quad \text{If } a \subseteq C(X) \text{ is an ideal, then } \sigma(a) = \text{compatibility}(a, a) \text{ is the } \mathbf{z}\text{-ideal generated by } a. \]

\[ \text{Proposition.} \quad \text{If } a \subseteq C(X) \text{ is an ideal, then } \sigma(a) \text{ is of the form } \text{compatibility}(b, b) \text{ for some ideal } b \subseteq C(X). \]

\[ \text{Proof.} \quad \text{Let } a \subseteq C(X) \text{ be an ideal. Then \sigma(a) is the smallest}\]
Proof. (i) holds, since $C(X)$ is a real closed ring and the minimal points in Spec $C(X)$ are $z$-radical.

(ii). Take $q \subseteq p$ minimal over $a$. By (i) and since $\sqrt{a} \subseteq p^\circ \subseteq p$ the minimality of $p$ in $V(\sqrt{a})$ implies that $p$ is $z$-radical. Thus $\sqrt{a} \subseteq \sqrt{q} \subseteq p$, hence $\sqrt{q} = p$.

Since $C(X)$ is real closed we know that $q + \sqrt{a} \subseteq p$ is prime, hence $p = q + \sqrt{a}$.

(iii). If $p \in V(a)_{\text{min}}$, then choose $q \in V(a^\circ)_{\text{min}}$ with $q \subseteq p^\circ$. Then $q + \sqrt{a}$ is a prime ideal contained in $p$, so $q + \sqrt{a} = p$.

(iv) follows from (ii) and (iii) by applying (3.7). 

\[ \Box \]

Proposition. If $(a_i)_{i \in I}$ is an arbitrary family of ideals of $C(X)$, then

\[ \bigcap_{i \in I} a_i^\circ = \left( \bigcap_{i \in I} a_i \right)^\circ. \]

Proof. \( \supseteq \) holds, since the diamond operation is monotone. \( \subseteq \) holds, since \( \bigcap_{i \in I} a_i^\circ \) is a $z$-radical ideal contained in \( \bigcap_{i \in I} a_i \), thus \( \bigcap_{i \in I} a_i^\circ \subseteq \left( \bigcap_{i \in I} a_i \right)^\circ. \)

Example. For any $p \in \text{Spec } C(X)$, the ideal $\sqrt{p}$ is prime, since it is radical and it contains the prime ideal $p$.

The converse is not true! That is, if $a$ is a radical ideal of $C(X)$ s.t. $\sqrt{a}$ is prime, then $a$ need not be prime, even if $a$ is finitely generated as a radical ideal. For example, if $X = \mathbb{R}$ and $a = \sqrt{x}$, where $x$ is the identity on $\mathbb{R}$, then $\sqrt{a} = \sqrt{0} = 0$, which is even maximal.

On the other hand, $a$ is not prime. To see this, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 0$ if $x \leq 0$ and $f(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} \sqrt{x \wedge 1}$ if $x \geq 0$. Then $f$ is continuous and an easy calculation shows that no power of $f$ is a multiple of $x$ in $C(\mathbb{R})$. Hence $f \not\in a$. Then also $f(-x) \not\in a$. But $f(x)f(-x) = 0 \in a$, which shows that $a$ is not prime.

A branching point of Spec $C(X)$ is a prime ideal $p$ of $C(X)$, which is not minimal, such that for each proper generalization $q$ of $p$, there is a generalization of $p$, which is incomparable with $q$. By [Gil-Jer], Section 14, all branching points of Spec $C(X)$ are $z$-radical. We need this fact in a slightly more general version:

Proposition. All branching points of Spec $C(X)$ are $z$-radical. More precisely, if $p, q \in \text{Spec } C(X)$ are incomparable, then

\[ p + q = p^\circ + q^\circ = \sqrt{p} + \sqrt{q} \]

is $z$-radical.

Proof. Since $C(X)$ is real closed we know that $p^\circ + q^\circ$ is a prime ideal or equal to $C(X)$. Moreover $p$ and $p^\circ + q^\circ$ are comparable and $q$ and $p^\circ + q^\circ$ are comparable. Since $p$ and $q$ are incomparable, we can not have $p^\circ + q^\circ \subseteq p, q$. Say $p \subseteq p^\circ + q^\circ$. Then $p^\circ + q^\circ$ can not be contained in $q$. Hence $q \subseteq p^\circ + q^\circ$ and $p + q \subseteq p^\circ + q^\circ$ which shows that $p + q = p^\circ + q^\circ$ is $z$-radical. But then clearly $\sqrt{p}, \sqrt{q} \subseteq p + q$ as claimed. \( \Box \)
It follows that \((p + q)^\circ = p^\circ + q^\circ\) for all prime ideals \(p, q\) of \(C(X)\) and one might ask if this additivity also holds for all radical ideals of \(C(X)\). In (9.14) below we show that this is not the case, even if \(X\) is compact. However, if \(X\) is compact or a subset of \(\mathbb{R}^n\) and \(a, b\) are radical ideals of \(C(X)\) which are finitely generated as radical ideals, then \((a + b)^\circ = a^\circ + b^\circ\) (cf. (9.12)).

4. Tubular ideals in rings with normal spectrum

First an excursion about tubular neighborhoods of subsets of spectral spaces.

(4.1) Definition. Let \(X\) be an arbitrary topological space and let \(Z \subseteq X\). We define

\[
\theta(Z) := \bigcap_{Z \subseteq O \subseteq X, O \text{ open}} O.
\]

(4.2) Lemma. Let \(X\) be an arbitrary topological space.

(i) For every subset \(Z\) of \(X\), \(\theta(Z)\) is the set of all generalizations of points from \(Z\):

\[
\theta(Z) = \{ x \in X \mid x \twoheadrightarrow z \text{ for some } z \in Z \}.
\]

(ii) For every collection \(\{Z_i \mid i \in I\}\) of subsets \(Z_i\) of \(X\) we have

\[
\theta\left(\bigcup_{i \in I} Z_i\right) = \bigcup_{i \in I} \theta(Z_i) \quad \text{and} \quad \theta\left(\bigcap_{i \in I} Z_i\right) \subseteq \bigcap_{i \in I} \theta(Z_i).
\]

Proof. (i) is straightforward and left to the reader.

\[
\text{LONG VERSION} = \begin{array}{c}
\text{proof of } \supseteq. \text{ Let } x \notin \text{Gen}(Z). \text{ Hence for each } z \in Z \text{ there is an open set } U_z \text{ with } x \notin U_z \ni z. \\
\text{Hence } O := \bigcup_{z \in Z} U_z \text{ is an open set containing } Z \text{ and not containing } x, \text{ i.e. } x \notin \theta(Z).
\end{array}
\]

\[
\text{END OF LONG VERSION} = \]

(ii) is an obvious consequence of (i). \(\square\)

(4.3) Lemma. Let \(X\) be a spectral space.

(i) If \(Z \subseteq X\) is quasi-compact, then \(Z^{\text{max}}\) is again quasi-compact and

\[
\theta(Z^{\text{max}}) = \theta(Z) = \bigcap_{Z \subseteq O \subseteq K(X)} O
\]

is the closure \(Z^{\text{opp}}\) of \(Z\) in the inverse topology of \(X\).

(ii) If \(X\) is a normal spectral space then for every collection \(\{Z_i \mid i \in I\}\) of closed subsets \(Z_i\) of \(X\) we have

\[
\theta\left(\bigcap_{i \in I} Z_i\right) = \bigcap_{i \in I} \theta(Z_i).
\]

Proof. (i). First we show that \(Z \subseteq \theta(Z^{\text{max}})\). Let \(Z_0 \subseteq Z\) be totally ordered by \(\twoheadrightarrow\). Since \(Z\) is quasi-compact, there is some \(z \in Z\) inside \(\bigcap_{z_0 \in Z_0} \{z_0\}\). Hence by Zorn’s lemma (and (4.2)(i)) we get \(Z \subseteq \theta(Z^{\text{max}})\).

It follows \(\theta(Z^{\text{max}}) = \theta(Z)\) and every open covering of \(Z^{\text{max}}\) is also an open covering of \(Z\), thus \(Z^{\text{max}}\) is quasi-compact, too.

By (2.7)(i) applied to the inverse topology of \(X\), the closure \(Z^{\text{opp}}\) of \(Z\) in the inverse topology of \(X\) is the set of all generalizations of points from \(Z\), hence \(\theta(Z) = Z^{\text{opp}}\).
(ii). By (4.2)(ii) it is enough to show $\mathcal{O}(\bigcap_{i \in I} Z_i) \supseteq \bigcap_{i \in I} \mathcal{O}(Z_i)$. If $x \in \mathcal{O}(Z_i)$ for every $i \in I$, then by (4.2)(i), $x \to z_i$ for some $z_i \in Z_i$ ($i \in I$). Since the $Z_i$ are closed, we may assume that $z_i \in X^{\max}$. Since $X$ is normal, $z_i = z_j$ for all $i, j \in I$. Hence $z_i \in \bigcap_{i \in I} Z_j$, thus $x \in \mathcal{O}(\bigcap_{i \in I} Z_j)$.

(4.4) **Lemma.** Let $A$ be a ring and let $Z \subseteq \text{Spec } A$ be closed under specializations. Then

(i) If every $p \in \text{Spec } A$ we have

$$p \in \mathcal{O}(Z) \Leftrightarrow 1 \notin p + q \text{ for some } q \in Z.$$

(ii) If $Z = V(a)$ for some ideal $a$ of $A$, then

$$p \in \mathcal{O}(Z) \Leftrightarrow 1 \notin p + a.$$

**Proof.** (i) is straightforward, we prove (ii).

---

(4.5) **Definition.** If $A$ is a ring and $a$ is an ideal of $A$, then we define

$$O(a) := \bigcap \mathcal{O}(V(a)).$$

We call $O(a)$ the **tubular ideal** of $a$.

Hence $O(a)$ is a radical ideal with $V(a) \subseteq \mathcal{O}(V(a)) \subseteq V(O(a)) = \overline{\mathcal{O}(V(a))}$, in particular $O(a) = O(\sqrt{a}) \subseteq \sqrt{a}$. Moreover $O(a) \subseteq O(b)$ for all ideals $a \subseteq b$ of $A$.

If $\text{Spec } A$ is normal, then $\mathcal{O}(V(a))$ is closed, hence $\mathcal{O}(V(a)) = V(O(a))$ and $\mathcal{O}(V(a))^{\max} = V(a)^{\max}$; therefore $O(a) = O(O(a))$ for all ideals $a$ of such a ring.

(4.6) **Lemma.** Let $X$ be a spectral space and let $K \subseteq X$ such that $X \setminus K$ is quasi-compact in the inverse topology of $X$. Let $x \in X$. Then $x \in \text{int}(K)$ if and only if $\mathcal{O}(x) \subseteq K$.

**Proof.** Clearly $x \in \text{int}(K)$ implies $\mathcal{O}(x) \subseteq K$. Conversely, if $\mathcal{O}(x) \subseteq K$, then $\bigcap_{x \in U \subseteq K} U \subseteq K$ and as $X \setminus K$ is quasi-compact in the inverse topology of $X$, there must be some $x \in U \subseteq \text{int}(K)$ with $U \subseteq K$.

(4.7) **Corollary.** Let $A$ be a ring and let $Y \subseteq \text{Spec } A$. Then

$$\{ f \in A \mid Y \subseteq \text{int } V(f) \} = \bigcap \mathcal{O}(Y).$$

In particular

$$O(a) = \{ f \in A \mid V(a) \subseteq \text{int } V(f) \}$$

for every ideal $a$ of $A$.

**Proof.** We have $Y \subseteq \text{int } V(f)$ if and only if $p \in \text{int } V(f)$ for all $p \in Y$, if and only if $f \in \bigcap \mathcal{O}(p)$ for all $p \in Y$ (by (4.6)), if and only if $f \in \bigcap \mathcal{O}(Y)$ (by (4.2)(ii)).

If we apply this assertion to $Y = V(a)$ we get $O(a) = \bigcap \mathcal{O}(V(a)) = \{ f \in A \mid V(a) \subseteq \text{int } V(f) \}$. 

□
If we apply (4.3)(ii) to Spec $A$ we get

(4.8) Corollary. Let $a, b$ be ideals of a ring $A$. Then $O(a \cdot b) = \sqrt{O(a) \cdot O(b)}$. If $Spec A$ is normal, then $O(a + b) = \sqrt{O(a) + O(b)}$.

Proof. Since $V(O(ab)) = \overline{O(V(ab))}$ it is enough to show that $\overline{O(V(ab))} = V(O(a)O(b))$. We have $\overline{O(V(ab))} = \overline{O(V(a) \cup V(b))} = \overline{O(V(a)) \cup \overline{O(V(b))}} = \overline{O(V(a))} \cup \overline{O(V(b))} = V(O(a)O(b))$.

Now suppose $Spec A$ is normal. It is enough to show that $V(O(a + b)) = V(O(a) + O(b))$. We have $V(O(a + b)) = \overline{O(V(a + b))} = \overline{O(V(a) \cap V(b))} = \overline{O(V(a))} \cap \overline{O(V(b))}$ by (4.3)(ii). Hence $V(O(a + b)) = \overline{O(V(a))} \cap \overline{O(V(b))} = V(O(a) + O(b))$ as desired. \hfill \Box

(4.9) Proposition. If $a$ is an ideal of a ring $A$, then

$O(a) = \{ f \in A \mid \text{there are } a \in a \text{ and } k \in \mathbb{N} \text{ such that } f^k \cdot (1 - a) = 0 \}$

is the radical ideal generated by the kernel of the localization map $A \longrightarrow (1 + a)^{-1}A$. In particular $O(a) = \{ f \in A \mid \exists a \in a : f = f \cdot a \} \subseteq a$ if $A$ is a reduced ring.

Proof. If $f^k \cdot (1 - a) = 0$, $a \in a$, then $V(a) \subseteq D(1 - a) \subseteq V(f)$, hence $f \in O(a)$ by (4.7). Conversely if $f \in O(a)$, then by (4.7) we have $V(a) \subseteq \text{int}(V(f))$. Since $V(a)$ is quasi-compact, there are $g_1, \ldots, g_n \in A$ with $V(a) \subseteq D(g_1) \cup \ldots \cup D(g_n) \subseteq V(f)$. The inclusion $D(g_i) \subseteq V(f)$ says that $g_i f$ is nilpotent, hence there is some $k \in \mathbb{N}$ such that $g_i^k f^k = 0$ for all $i \in \{1, \ldots, n\}$. The inclusion $V(a) \subseteq D(g_1) \cup \ldots \cup D(g_n)$ says that $1 \in a + (g_1^k, \ldots, g_n^k)$ and there are $h_1, \ldots, h_n \in A$, $a \in a$ with $1 - a = h_1 g_1^k + \ldots + h_n g_n^k$. If we multiply this equation with $f^k$ we get $f^k \cdot (1 - a) = f^k \cdot (h_1 g_1^k + \ldots + h_n g_n^k) = 0$. \hfill \Box

(4.10) Remark. Let $a, b$ be ideals of a ring $A$. We define

$a : b := \{ f \in A \mid f \cdot b \subseteq a \}$

Hence $a : b$ is equal to the annihilator ideal $\text{ann}((b + a)/a)$ of $A$.

We have

$\sqrt{a : b} = \bigcap V(a) \setminus V(b)$,

in other words $\sqrt{a : b}$ is a radical ideal and

$V(\sqrt{a : b}) = V(a) \setminus V(b)$.

In particular, if $A$ is a reduced ring, then $\text{ann} b$ is a radical ideal with

$V(\text{ann} b) = \text{Spec } A \setminus V(b)$.

Proof. Let $I := \bigcap V(a) \setminus V(b)$, hence $V(I) = V(a) \setminus V(b)$. Then for each $r \in A$ we have $r \in \sqrt{a : b} \iff r \cdot b \subseteq \sqrt{a} \iff V(a) \subseteq V(rb)$. Since $V(rb) = V(r) \cup V(b)$ we get $r \in \sqrt{a : b} \iff V(a) \subseteq V(r) \cup V(b) \iff V(a) \setminus V(b) \subseteq V(r) \iff V(a) \setminus V(b) \subseteq V(r) \iff V(r) \subseteq I$. \hfill \Box

(4.11) Corollary. Let $a$ be an ideal of a reduced ring $A$. Then

(i) $O(a) = \{ f \in A \mid f + \text{ann } f = A \} \subseteq a$

(ii) If $A$ is reduced and every open quasi-compact subset of $A$ is of the form $D(g)$ for some $g \in A$ (e.g. if $A$ is real closed), then

$O(a) = \{ f \in A \mid \exists a \in a \text{ with } f = f \cdot a \}$. 
In particular \( a \cdot O(a) = O(a) \).

**Proof.** (i). Let \( f \in A \). Since \( A \) is reduced, the annihilator \( \text{ann} f \) of the ideal \( f \cdot A \) is the defining ideal of \( \text{Spec } A \setminus \text{int } V(f) \). Then \( V(a) \subseteq \text{int } V(f) = \text{Spec } A \setminus \text{ann } f \) is equivalent to \( a + \text{ann } f = A \). By (4.7) we get the equality in (i).

It follows \( O(a) \subseteq a \), since for \( a \in A, b \in A \) with \( 1 = a + b \) and \( f \cdot b = 0 \) we have \( f = f \cdot 1 = f \cdot a \in a \).

(ii). The inclusion \( \supseteq \) holds without any assumption on \( A \): if \( f = f \cdot a \), \( a \in a \), then \( V(a) \subseteq D(1 - a) \subseteq V(f) \).

For the converse we assume that \( A \) is reduced and that every open quasi-compact subset of \( A \) is of the form \( D(g) \) for some \( g \in A \). Then, if \( f \in O(a) \) we know that there is some \( g \in A \) with \( V(a) \subseteq D(g) \subseteq V(f) \). Hence \( g \cdot f = 0 \) (as \( A \) is reduced) and \( a + g \cdot A = A \). Pick \( b \in A \) and \( a \in a \) with \( 1 = a + b \). Then \( f \cdot (1 - a) = f \cdot (a + b) - f \cdot b = f b g = 0 \). \( \square \)

=-------------------------- END OF LONG VERSION =--------------------------

(4.12) **Definition.** Let \( a \) be an ideal of a ring \( A \). We define

\[
m\sqrt[\text{max}]{a} := \bigcap V(a)^\text{max}.
\]

Hence \( m\sqrt[\text{max}]{a} \) is the Jacobson radical of \( a \). Recall from commutative algebra that

\[
m\sqrt[\text{max}]{a} = \{ f \in A \mid \forall x \in A : 1 \in a + (1 + x \cdot f) \}.
\]

(4.13) **Proposition.** Let \( a \) be an ideal of a ring \( A \). Then

(i) \( \sqrt[\text{max}]{a} = \{ f \in A \mid V(a) \subseteq \mathcal{O}(V(f)) \} = \{ f \in A \mid \mathcal{O}(V(a)) \subseteq \mathcal{O}(V(f)) \} \) and \( \mathcal{O}(\sqrt[\text{max}]{a}) = O(a) \).

(ii) If \( A \) is reduced and \( b \) is an ideal of \( A \) with \( a \subseteq m\sqrt[\text{max}]{b} \), then \( O(a) \subseteq b \).

**Proof.** (i). If \( V(a) \subseteq \mathcal{O}(V(f)) \), then every \( m \in V(a) \) specializes to a point in \( V(f) \), i.e. \( m \in V(f) \) and this shows \( f \in m\sqrt[\text{max}]{a} \).

Conversely suppose \( V(a) \subseteq \mathcal{O}(V(f)) \). Pick \( p \in V(a) \setminus \mathcal{O}(V(f)) \). Thus \( p \) specializes to no point in \( V(f) \), i.e. \( 1 \in p + f \cdot A \). Hence \( 1 \in m + f \cdot A \) for every \( m \in V(p) \) as \( V(a) \) as well, which implies that \( f \notin m\sqrt[\text{max}]{a} \).

Since \( \mathcal{O}(V(a)) \subseteq \mathcal{O}(V(f)) \) is equivalent to \( V(a) \subseteq \mathcal{O}(V(f)) \) we get the first equalities.

In order to see \( \mathcal{O}(\sqrt[\text{max}]{a}) = O(a) \) it is enough to show \( \mathcal{O}(\mathcal{O}(\sqrt[\text{max}]{a})) = \mathcal{O}(\sqrt[\text{max}]{a}) \). We certainly have \( V(\sqrt[\text{max}]{a}) = V(a)^\text{max} \) and we know \( \mathcal{O}(Z) = \mathcal{O}(Z^\text{max}) \) for all quasi-compact subsets \( Z \) of \( \text{Spec } A \) by (4.2). Therefore \( \mathcal{O}(V(a)) = \mathcal{O}(V(\sqrt[\text{max}]{a})) = \mathcal{O}(\mathcal{O}(\sqrt[\text{max}]{a})) \) as desired.

(ii). Let \( a \subseteq m\sqrt[\text{max}]{b} \) and take \( f \in O(a) \). By (4.11) there is some \( a \in a \) with \( f = f \cdot a \). Since \( a \in a \subseteq m\sqrt[\text{max}]{b} \) we have \( 1 \in b + (1 - a) \). Hence \( 1 = b + y \cdot (1 - a) \) for some \( y \in A \) and some \( b \in b \). Multiplying with \( f \) yields \( f = f b + y \cdot (f - fa) = f b + b \). \( \square \)

(4.14) **Corollary.** If \( A \) is reduced and \( \text{Spec } A \) is normal, then

\[
m\sqrt[\text{max}]{a} = \{ f \in A \mid O(f) \subseteq a \} = \{ f \in A \mid O(f) \subseteq O(a) \} \text{ and } O(a) \text{ is the smallest ideal } b \text{ of } A \text{ with } a \subseteq m\sqrt[\text{max}]{b}.
\]

The map

\[
\{ a \subseteq C(X) \mid a = m\sqrt[\text{max}]{a} \} \longrightarrow \{ b \subseteq C(X) \mid b = O(b) \}
\]

\[
a \longmapsto O(a)
\]

is a bijection and the inverse map sends an ideal \( b \) to \( m\sqrt[\text{max}]{b} \).
The interest in the ideals map as claimed.

In particular $O(\mathfrak{a}) = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of $A$. Therefore the first assertion follows from (4.13),(i).

Since $\mathfrak{a}$ is quasi-compact for a lot of ideals $\mathfrak{a}$ of $C(X)$ we get a bijective map as claimed.

If $Y$ is a dense subset of an arbitrary topological space $X$, then for every closed subset $A$ of $X$, the interior of $A \cap Y$ with respect to $Y$ is the set of interior points of $A$ contained in $Y$.

Proof. If $y \in \text{int}_Y(A \cap Y)$, then there is some $O \subseteq X$ open with $y \in O \cap Y \subseteq A \cap Y$. Since $Y$ is dense in $X$ there is no point in $O \setminus A$. Hence $O \subseteq A$ and $y$ is in the interior of $A$. □

Therefore

(4.15) Lemma. Let $A$ be a ring such that $(\text{Spec } A)_{\text{max}}$ is dense in Spec $A$ and let $\mathfrak{a}$ be an ideal of $A$. Then

$$O(\mathfrak{a}) = \{ f \in A \mid V(\mathfrak{a})_{\text{max}} \subseteq \text{int}_{(\text{Spec } A)_{\text{max}}} \{ V(f)_{\text{max}} \} \}.$$

Proof. Since $V(f)_{\text{max}} = V(f) \cap (\text{Spec } A)_{\text{max}}$ and $(\text{Spec } A)_{\text{max}}$ is dense in Spec $A$, the foregoing remark says that $V(\mathfrak{a})_{\text{max}} \subseteq \text{int}_{(\text{Spec } A)_{\text{max}}} \{ V(f)_{\text{max}} \}$ is equivalent to $V(\mathfrak{a})_{\text{max}} \subseteq \text{int } V(f)$.

If we weaken the condition $\{ V(\mathfrak{a})_{\text{max}} \subseteq \text{int}_{\beta X} V(f)_{\text{max}} \}$ of (4.15) in the case $\mathfrak{a} = (g)$ to $\{ \text{int } f = 0 \}$, then we'll get $f \in O(g)$ in (9.2) below for many topological spaces $X$ (e.g. locally compact spaces or metric spaces).

The generic tubular ideal $GO(\mathfrak{a})$

This section is not used in the sequel.

(4.16) Remark. Let $\mathfrak{a}$ be an ideal of a ring $A$. We define

$$GO(\mathfrak{a}) := \{ f \in A \mid V(\mathfrak{a})_{\text{min}} \subseteq \text{int } V(f) \}.$$

Since $V(f \cdot g) = V(f) \cup V(g)$ and $\text{int } V(f) \cap \text{int } V(g) = \text{int } (V(f) \cap V(g) \subseteq \text{int } V(f + g)$,

$GO(\mathfrak{a})$ is indeed a radical ideal of $A$.

Moreover, $GO(\mathfrak{a}) \subseteq \sqrt{\mathfrak{a}}$, since $V(\mathfrak{a})_{\text{min}} \subseteq \text{int } V(f)$ implies $V(\mathfrak{a}) \subseteq V(f)$.

By (4.7) we know that $O(\mathfrak{a}) \subseteq GO(\mathfrak{a})$. Moreover by (4.7) we know that

$$GO(\mathfrak{a}) = \bigcap_{q \in V(\mathfrak{a})_{\text{min}}} q.$$

In particular $O(\mathfrak{a}) \not\subseteq GO(\mathfrak{a})$ in general. We have $V(GO(\mathfrak{a})) = \overline{\text{Gen } V(\mathfrak{a})_{\text{min}}} \subseteq \overline{\mathfrak{a}}(V(\mathfrak{a})_{\text{min}})$.

The interest in the ideals $GO(\mathfrak{a})$ stems from the fact that $V(\mathfrak{a})_{\text{min}}$ is quasi-compact for a lot of ideals $\mathfrak{a}$ of real closed rings. We'll not use these ideals in this paper.
The interest in the generic tubular ideals stems from the fact that $\text{Spec } A$ is equivalent to $\text{int } V$. If $\text{max}$ and $\text{GO}$ then $\text{GO}(a)$ is indeed a radical ideal of $A$, since $V(a)^{\text{min}} \subseteq \text{int } V(a)$. Moreover, $\text{GO}(a) \subseteq \sqrt{a}$, since $V(a)^{\text{min}} \subseteq \text{int } V(a)$ implies $V(a) \subseteq V(f)$.

By (4.7) we know that $O(a) \subseteq \text{GO}(a)$. Moreover by (4.7) we know that

$$\text{GO}(a) = \bigcap_{q \subset p \in V(a)^{\text{min}}} q.$$

In particular $O(a) \subsetneq \text{GO}(a)$ in general.

We have $V(\text{GO}(a)) = \text{Gen } V(a)^{\text{min}} \subseteq \mathcal{O}(V(a)^{\text{min}})$.

Let $f \in A$ and let $b$ be the defining ideal of $\text{Spec } A \setminus \text{int } V(f)$. Then $\text{Spec } A \setminus V(b) \subseteq V(f)$ is equivalent to $f \cdot b = 0$ (here we need that $A$ is reduced). Moreover $V(a)^{\text{min}} \subseteq \text{int } V(f) = \text{Spec } A \setminus V(b)$ is equivalent to $V(a)^{\text{min}} \cap V(b) = \emptyset$.

Suppose $V(a)^{\text{min}} \cap V(b) = \emptyset$. Then $V(a) \setminus V(b)$ is dense in $V(a)$.

The interest in the generic tubular ideals stems from the fact that $V(a)^{\text{min}}$ is quasi-compact for a lot of ideals $a$ of real closed rings. In this case, $V(a)^{\text{min}}$ and $\text{Gen } V(a)^{\text{min}}$ are even proconstructible. (cf. minimalcompact:(17) and characterize quasi-compact subsets) and $\text{Gen } V(a)^{\text{min}} = \mathcal{O}(V(a)^{\text{min}})$.

(4.18) **LEMMA.** Suppose that $V(a)^{\text{min}}$ is quasi-compact. Then

$$V(\text{GO}(a)) = \mathcal{O}(V(a)^{\text{min}})$$

and $\text{GO}(a) = \emptyset$.

**PROOF.** Since $V(a)^{\text{min}}$ is quasi-compact, it is proconstructible and it contains the maximal points of its closure $Z$ in the constructible topology.

**UNDER CONSTRUCTION:**

\[\square\]

We return to $C(X)$ and we first summarize, what the general theory gives us so far.

(4.19) **SUMMARY.** Since $C(X)$ is a real closed ring and $\beta X$ is dense in $\text{Spec } C(X)$ we have for every ideal $a$ of $C(X)$:

1. $O(a) = \bigcap_{p \in V(O(a)) \cap \text{Spec } C(X)^{\text{min}}} p$ is a $z$-radical ideal contained in $a$, hence $O(a) \subseteq a^{\circ}$.
2. $\mathcal{O}(V(a)) = V(O(a))$ is closed and closed in the inverse topology of $\text{Spec } C(X)$.
3. $\sqrt{a} = \bigcap V(a)^{\text{max}}$ is $z$-radical, containing $\sqrt{a}$. Hence we have $O(a) \subseteq a^{\circ} \subseteq a \subseteq \sqrt{a} \subseteq \sqrt{a}$.
4. For every ideal $b$ of $C(X)$ with $O(a) \subseteq b \subseteq \sqrt{a}$ we have $O(a) = O(b)$, $V(b) \cap \beta X = V(a) \cap \beta X$ and $\sqrt{a} = \sqrt{b}$.
5. If $b$ is another ideal of $C(X)$, then $O(a \cdot b) = \sqrt{O(a) \cdot O(b)}$ and $O(a + b) = O(a) + O(b)$.
6. If $f \in C(X)$, then the following are equivalent.
   - (i) $f \in O(a)$.
   - (ii) $V(a) \subseteq \text{int } V(f)$.
(iii) $V(a)^{\text{max}} \subseteq \text{int}_{\beta X} V(f)^{\text{max}}$.

(iv) $\exists a : f = f \cdot a$.

7. $\sqrt{a} = \bigcup_{f \in a} {\text{max}}(\sqrt{f})$, in particular $\sqrt{f} = {\text{max}}(\sqrt{f})$ for all $f \in C(X)$. This is so since for every $f \in C(X)$, by definition of $\sqrt{f}$ we have $\sqrt{f} = \bigcap_{x \in (f = 0)} x$, thus $\sqrt{f} \subseteq {\text{max}}(\sqrt{f}) \subseteq \bigcap_{x \in (f = 0)} x = \sqrt{f}$.

--- LONG VERSION ---

Remark. In (9.2) below we prove $O(f) = (f)^{\alpha}$ for many topological spaces $X$ and all $f \in C(X)$. For arbitrary ideals $a$ of $C(X)$, the ideal $a^{\alpha}$ differs dramatically from $O(a)$, since $f \in a^{\alpha}$ does not imply $V(a) \cap \beta X \subseteq \text{int}_{\beta X}(V(f) \cap \beta X)$: For example take $a = \hat{0} \in \beta \mathbb{R}^{\beta}$; then the norm function $|x|$ is in $a = a^{\alpha}$, but $V(a) \cap \beta \mathbb{R}^{\beta} = V(|x|) \cap \beta X = \{0\}$, which is not open in $\beta \mathbb{R}^{\beta}$.

Nevertheless, even the difference between $O(a)$ and $\text{max}(\sqrt{a})$ is invisible in $\beta X$, as $V(O(a)) \cap \beta X = V(\text{max}(\sqrt{a})) \cap \beta X$.

Observe also, that $V(a^{\alpha})$ is not the intersection of open sets, for example if $a$ is a prime $\mathbb{Z}$-ideal that is not minimal!

Finally we see that for $m \in \beta X$, the ideal $O(m)$ is equal to the Gillman-Jerison "$O^{\mu}$", defined in 7.12 of [Gil-Jer].

--- END OF LONG VERSION ---

(4.20) Definition. Let $a$ be an ideal of $C(X)$. We say that $a$ is \textbf{geometrically simple} if $\sqrt{a} = \sqrt{f}$ for some $f \in C(X)$.

In other words, an ideal $a$ is geometrically simple, if and only if $V(a) \cap X = \{f = 0\}$ for some $f \in a$. Observe that $a$ is geometrically simple if $\sqrt{a} = \sqrt{f_{1}, ..., f_{n}}$ for some $f_{1}, ..., f_{n} \in C(X)$, since $\sqrt{f_{1}, ..., f_{n}} = \sqrt{f_{1}^{2} + ... + f_{n}^{2}}$.

--- END OF LONG VERSION ---

(4.21) Proposition. For every ideal $a$ of $C(X)$ we have $O(a) = \{g \in C(X) \mid \exists f \in a, h \in C(X) : \{f = 0\} \subseteq \{h \neq 0\} \subseteq \{g = 0\}\}$.

Proof. If $g \in O(a)$, then $V(a) \subseteq \text{int} V(g)$, by (4.7). Since $V(a)$ is quasi-compact, there is some $h \in C(X)$ with $V(a) \subseteq D(h) \subseteq V(g)$. By (2.6), there is some $f_{1} \in C(X)$ with $V(a) \subseteq V(f_{1}) \subseteq D(h)$. Then $f_{1} \in a$ and for some $n \in \mathbb{N}$, $f := f_{1}^{n}$ satisfies $f \in a$ and $V(f) \subseteq D(h) \subseteq V(g)$; the latter inclusions imply $\{f = 0\} \subseteq \{h \neq 0\} \subseteq \{g = 0\}$.

Conversely suppose $g, h \in C(X)$ and $f \in a$ with $\{f = 0\} \subseteq \{h \neq 0\} \subseteq \{g = 0\}$. Thus $h \cdot g = 0$ and $D(h) \subseteq V(g)$. On the other hand $V(f) \subseteq D(h)$ (otherwise $V(f^{2} + h^{2}) \supseteq V(f) \cap V(h) \neq 0$, which contradicts $f^{2} + h^{2} > 0$).

This shows $V(a) \subseteq V(f) \subseteq D(h) \subseteq V(g)$, which means $g \in O(a)$.

(4.22) Remark. Observe that for a normal space $X$ and $f, g \in C(X)$ with $\{f = 0\} \subseteq \text{int}(g = 0)$ there is always a cozero set $O$ of $X$ with $\{f = 0\} \subseteq O \subseteq \text{int}(g = 0)$.

(4.23) Corollary. Let $f \in C(X)$ and let $p \in \text{Spec} C(X)$ with $1 \notin p + (f)$. Then for every $g \in C(X)$ the following hold true:

(i) If there is some $h \in C(X)$ with $\{f = 0\} \subseteq \{h > 0\} \subseteq \{g \geq 0\}$, then $g \mod p \geq 0$. 

\[\text{Proof.} \ \quad \]
(ii) If \( \{ f = 0 \} \subseteq \{ g > 0 \} \), then \( g \mod p > 0 \).

**Proof.** (i). We have \( \{ g \geq 0 \} = \{ g^- = 0 \} \). Since \( \{ h > 0 \} \) is the cozero set \( \{ h^+ \neq 0 \} \), the assumption \( \{ f = 0 \} \subseteq \{ h > 0 \} \subseteq \{ g \geq 0 \} \) together with (4.21) implies that \( g^- \in O(f) \).

Since \( 1 \not\in p + (f) \) we know \( p \in \mathcal{O}(V(f)) \) by (4.4)(ii). Since \( \mathcal{O}(V(f)) = V(O(f)) \) we get \( g^- \in O(f) \subseteq p \) and this means \( g \mod p > 0 \).

(ii). If \( g \not\in p \), then \( g^2 + f^2 \in p + (f) \). Since \( \{ f = 0 \} \subseteq \{ g > 0 \} \), \( g^2 + f^2 \) does not have zeroes, so \( 1 \in p + (f) \), in contradiction to our assumption.

Hence \( g \not\in p \) and from (i) we get \( g \mod p > 0 \). \( \square \)

In the rest of this section we characterize the ideals \( O(a) \) and \( \sqrt[n]{a} \) for ideals \( a \) of \( C(X) \) in terms of inequalities.

Let \( \varepsilon \in C(X) \), \( \varepsilon > 0 \) everywhere. The following Proposition says that each sandwich \( Z_1 \subseteq D \subseteq Z_2 \) consisting of zero sets \( Z_i \) and a cozero set \( D \) can be realized as \( \{ e = 0 \} \subseteq \{ |e| < \varepsilon \} \subseteq \{ |e| \leq \varepsilon \} \) for some \( e \in C^*(X) \).

This allows a comparison of sets of the form \( \{ f = 0 \} \) with sets of the form \( \{|g| \leq \varepsilon \} \), from the ideal theoretic point of view (cf. (4.25))

**Proposition.** Let \( f, g, h \in C(X) \) with \( \{ f = 0 \} \subseteq \{ h \neq 0 \} \subseteq \{ g = 0 \} \). Let \( \varepsilon \in C(X) \), \( \varepsilon > 0 \) everywhere and let \( e \in C(X) \) be defined by

\[
e = \frac{f^2}{f^2 + h^2} \cdot \varepsilon + (1 + \varepsilon)g^2/1 + g^2 + h^2.
\]

Then \( 0 \leq \varepsilon \leq 1 + \varepsilon \), \( \{ e = 0 \} = \{ f = 0 \} \), \( \{ e < \varepsilon \} = \{ h \neq 0 \} \) and \( \{ e \leq \varepsilon \} = \{ g = 0 \} \).

**Proof.** Let \( \psi := \frac{\varepsilon + (1 + \varepsilon)g^2}{1 + g^2 + h^2} \). Then \( \psi \) does not have zeroes and

\[
\psi = \frac{\varepsilon(1 + g^2 + h^2) + (g^2 - \varepsilon h^2)}{1 + g^2 + h^2} = \varepsilon - \frac{\varepsilon h^2}{1 + g^2 + h^2} + \frac{g^2}{1 + g^2 + h^2}.
\]

We write \( \psi_1 := \varepsilon - \frac{\varepsilon h^2}{1 + g^2 + h^2} \) and \( \psi_2 := \frac{g^2}{1 + g^2 + h^2} \). Then \( 0 \leq \psi_1 \leq \varepsilon \) and \( 0 \leq \psi_2 \leq 1 \).

Since \( \{ \psi_1 < \varepsilon \} = \{ h \neq 0 \} \subseteq \{ g = 0 \} \) we get \( \{ \psi_1 + \psi_2 < \varepsilon \} = \{ h \neq 0 \} \) and then \( \{ \psi_1 + \psi_2 \leq \varepsilon \} = \{ g = 0 \} \). This shows

\[
(*) \quad 0 < \psi \leq 1 + \varepsilon, \quad \{ \psi < \varepsilon \} = \{ h \neq 0 \} \text{ and } \{ \psi \leq \varepsilon \} = \{ g = 0 \}.
\]

Let \( \varphi := \frac{f^2}{f^2 + h^2} \). Then \( \varphi \in C(X) \) is well defined with

\[
0 \leq \varphi \leq 1, \quad \{ \varphi = 0 \} = \{ f = 0 \} \text{ and } \{ \varphi = 1 \} = \{ h = 0 \}.
\]

Together with (*) this proves that \( e = \varphi \cdot \psi \) has the required properties. \( \square \)

**Corollary.** For every ideal \( a \) of \( C(X) \) and all \( g \in C(X) \) the following are equivalent.

(i) \( g \in O(a) \).

(ii) \( \forall e \in C(X), \varepsilon > 0 \) everywhere \( \exists f \in a : \{ |f| \leq \varepsilon \} = \{ g = 0 \} \).

(iii) \( \exists e \in C(X), \varepsilon > 0 \) everywhere \( \exists f \in a : \{ |f| \leq \varepsilon \} \subseteq \{ g = 0 \} \).

(iv) \( \exists f \in a, 0 \leq f \leq 1 \) with \( g = g \cdot f \).

(v) \( V(a) \cap \beta X \subseteq \text{int}_{\beta X} \{ g = 0 \}^{\beta X} \).

(vi) \( \forall e \in \mathbb{R}, \varepsilon > 0 \) \( \exists f \in C^*(X) : V(a) \cap \text{Spec} C(X) \subseteq V(f) \) and \( A \subseteq V(g) \), where \( A \) denotes the set of all prime \( z \)-ideals \( p \) such that \( \{ |f| \leq \varepsilon \} \) is a zero set of a function from \( p \).
Moreover, if there is some \( f \in a \) with \( a \subseteq \sqrt{(f)} \), then
\[
O(a) = \{ g \in C(X) \mid \exists \varepsilon \in C(X), \varepsilon > 0 \text{ everywhere with } \{ |f| \leq \varepsilon \} = \{ g = 0 \} \}.
\]

**Proof.** (i)\(\Rightarrow\)(ii). If \( g \in O(a) \), then by (4.21) there are \( f \in a \) and \( h \in C(X) \) with \( \{ f = 0 \} \subset \{ h \neq 0 \} \subset \{ g = 0 \} \). Take any \( \varepsilon \in C(X) \), \( \varepsilon > 0 \) on \( X \) and \( e \in C(X) \) as in (4.24) according to \( f, g, h \). By definition of \( e \) we get \( e \in (f) \subseteq a \) and \( \{|e| \leq \varepsilon \} = \{ g = 0 \} \).

(ii)\(\Rightarrow\)(iii) is a weakening.

(iii)\(\Rightarrow\)(i). Let \( \varepsilon \in C(X)^* \), \( \varepsilon > 0 \), \( g \in C(X) \) and \( f \in a \) with \( \{ |f| \leq \varepsilon \} \subset \{ g = 0 \} \). Then \( \{ f = 0 \} \subset \{ f^2 < \varepsilon^2 \} \subset \{ f^2 \leq \varepsilon^2 \} \subset \{ g = 0 \} \) and by (4.21) we see that \( g \in O(a) \).

So we know that (i)-(iii) are equivalent. By (4.11), item (iv) implies (i).

(ii)\(\Rightarrow\)(iv). Take \( f_0 \in a \) with \( \{|f_0| \leq 1 \} = \{ g = 0 \} \). Then \( f = \inf\{ f_0^2, 1 \} \) satisfies \( 0 \leq f \leq 1 \) and \( g = g \cdot f \). Since \( f \) and \( f_0^2 \) are equal on \( \{ f_0^2 < 1 \} \) we have \( f \in (f_0^2) \subseteq a \).

(i)\(\Leftrightarrow\)(v) holds by (4.15), since \( (g = 0)^{2X} = V(g) \cap \beta X \).

(vi)\(\Rightarrow\)(v). Let \( f \in C(X) \) and let \( A \) denotes the set of all prime \( z \)-ideals such that \( \{ |f| \leq 1 \} \) is a zero set of a function from \( p \). Then the set \( V(f) \cap \mathfrak{z} \operatorname{Spec} C(X) \) is contained in the interior of \( A \) with respect to \( z \)-\( \operatorname{Spec} C(X) \). In particular item (vi) implies item (v).

It remains to prove the moreover part. So we suppose there is some \( f \in a \) with \( a \subseteq \sqrt{(f)} \). Then if \( g \in O(a) \), there is \( h \in C(X) \) with \( \{ f = 0 \} \subset \{ h \neq 0 \} \subset \{ g = 0 \} \). Take again \( e \in C(X) \) as in (4.24) according to \( f, g, h \) with \( e = 1 \). So \( e = \frac{f^2}{f^2 + h^2} \cdot \frac{1 + 2 \sqrt{e}}{1 + 2 \sqrt{e}} \). Since \( \{ g = 0 \} = \{ e \leq 1 \} \) and \( e(x) \leq 1 \) if and only if \( |f(x)| \leq (\sqrt{(f^2 + h^2)} \cdot \frac{1 + 2 \sqrt{e}}{1 + 2 \sqrt{e}})(x) \) we can take \( e = \frac{\sqrt{(f^2 + h^2)} \cdot \frac{1 + 2 \sqrt{e}}{1 + 2 \sqrt{e}}}{\frac{1 + 2 \sqrt{e}}{1 + 2 \sqrt{e}}} \).

Observe that for \( f \in C(X) \) and \( g \in O(f) \) there need not be any \( \varepsilon \in \mathbb{R}, \varepsilon > 0 \) with \( \{|f| \leq \varepsilon \} \subseteq \{ g = 0 \} \). To see an example let \( X = \mathbb{R}^2 \), \( f(x, y) = y \) and let \( g \) be the distance function to \( \{(x, y) \in \mathbb{R}^2 \mid |xy| \leq 1 \} \).

(4.26) **Corollary.** For every ideal \( a \) of \( C(X) \) we have \( O(a) \cap C^*(X) = O(a \cap C^*(X)) \).

**Proof.** After identifying \( C^*(X) \) with \( C(\beta X) \) the assertion follows immediately from (4.25)(i)\(\Leftrightarrow\)(iv).

(4.27) **Lemma.** Let \( f, g \in C(X) \) and let \( \varepsilon, \delta \in C(X), \varepsilon, \delta > 0 \) everywhere. If there is a cozero set \( D \) of \( X \) with \( \{|f| \leq \varepsilon \} \subseteq D \subseteq \{|g| \leq \delta \} \), then there is some \( h \in O(f) \cap C^*(X) \) such that \( \{|g| \leq \delta \} = \{ |h| \leq \varepsilon \} \).

**Proof.** By (4.24), there is some \( h \in C^*(X) \), \( h \geq 0 \) such that \( \{|f| \leq \varepsilon \} = \{ h = 0 \} \) and \( \{|g| \leq \delta \} = \{ h \leq \varepsilon \} \). Since \( \{ f = 0 \} \subset \{|f| < \varepsilon \} \subset \{|h| = 0 \} \) we get \( h \in O(f) \) by (4.21).

(4.28) **Proposition.** If \( a \) is an ideal of \( C(X) \), then
\[
\underleftarrow{\varepsilon}a = \{ f \in C(X) \mid \forall \varepsilon \in C(X)^*, \varepsilon > 0 \exists h \in a : \{|f| \leq \varepsilon \} = \{|h| \leq \varepsilon \} \}.
\]

**Proof.** If \( \varepsilon \in C(X), \varepsilon > 0 \) everywhere and \( f \in C(X) \), then \( \{ f = 0 \} \subset \{|f| \leq \varepsilon \} \subset \{|f| < 2\varepsilon \} \subset \{|f| \leq 2\varepsilon \} \). By (4.27) there is some \( h \in O(f) \) such that \( \{|f| \leq \varepsilon \} = \{|h| \leq \varepsilon \} \). Hence, if \( f \in \underleftarrow{\varepsilon}a \) we get \( h \in a \) as desired.

Conversely take \( f \in C(X) \) such that for every \( \varepsilon \in C(X), \varepsilon > 0 \) everywhere there is some \( h \in a \) with \( \{|f| \leq \varepsilon \} = \{|h| \leq \varepsilon \} \). Let \( g \in O(f) \). By (4.25) there is some \( \varepsilon \in C(X), \varepsilon > 0 \)
everywhere, such that \(|f| \leq \varepsilon\) = \(|g| = 0\). By assumption, there is some \(h \in a\) with \(|f| \leq \varepsilon\) = \(|h| \leq \varepsilon\). By (4.25) again, \(g \in O(a)\). This shows that \(O(f) \subseteq O(a)\), thus \(f \in -\sqrt{a}\).

\(\square\)

Finally we express \(-\sqrt{a}\) for ideals of \(C^*(X)\) in terms of \(C(X)\):

(4.29) **PROPOSITION.** For every ideal \(a\) of \(C^*(X)\) we have

\[-\sqrt{a} = \{f \in C^*(X) \mid \forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \exists g \in a, \delta \in \mathbb{R}, \delta > 0 : \{|g| \leq \delta\} \subseteq \{|f| \leq \varepsilon\}\}.

**PROOF.** We use the isomorphism \(\varphi : C^*(X) \longrightarrow C(\beta X)\) which sends \(f\) to \(\beta f\).

If \(f \in -\sqrt{a}\) and \(\varepsilon \in \mathbb{R}, \varepsilon > 0\), then by (4.28) there is some \(g \in a\) such that \(|\beta g| \leq \varepsilon\). But then \(|g| \leq \varepsilon\) \(\subseteq\) \(|f| \leq \varepsilon\), too.

Conversely take \(f \in C^*(X)\) such that for all \(\varepsilon \in \mathbb{R}, \varepsilon > 0\) there is some \(g \in a\) with \(|g| \leq \delta\) \(\subseteq\) \(|f| \leq \varepsilon\). In order to prove \(f \in -\sqrt{a}\) we show \(\beta f \in -\sqrt{a}\) and we use (4.28) again. Take \(\varepsilon \in C(\beta X), \varepsilon > 0\) everywhere. We have to find some \(g \in a\) with \(|\beta g| \leq \varepsilon\).

Then \(|\beta g_1| \leq \delta\) \(\subseteq\) \(|\beta f| \leq \varepsilon_0\): take \(m \in \beta X\) with \(|\beta g_1|(m) \leq \delta\); then \(|\beta g_1| \geq 2\delta\) is not in the filter \(\mathfrak{f}\) of zero sets of \(X\) corresponding to \(m\); therefore \(|g_1| \geq 2\delta\) is in \(\mathfrak{f}\), thus \(|f| \leq \varepsilon_0\) \(\in \mathfrak{f}\) as well; this in turn means that \(|\beta f|(m) \leq \varepsilon_0\).

Finally let \(g\) be the restriction of \(h\) to \(X\). Then \(h = \beta g\), \(|\beta g| \leq \varepsilon\) = \(|\beta f| \leq \varepsilon\) and \(\beta g \in O(\beta g_1)\). In particular \(\beta g \in (\beta g_1)\) and \(g \in (g_1) \subseteq a\).

**PROPOSITION (4.29)** shows that \(-\sqrt{a}\) is the \(e\)-ideal generated by \(a\) in the sense of [Gil-Jer], 2L. Also at the same place in this book, \(e\)-filters are defined. By (4.25)(i)\(\Rightarrow\)(v) it is easy to see that the \(e\)-filters in the sense of [Gil-Jer] are precisely the filters \(\mathfrak{f}\) which satisfy \(\Theta(V(f)) = V(f)\) (in spectral space notation). We’ll not use this and leave the easy details to the reader.

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**ALTES MATERIAL, DASS JETZT SCHNELL BEWIESEN WERDEN KANN.**

(4.30) **COROLLARY.** Let \(A \subseteq \text{Spec } C(X)\) be closed and let \(V \subseteq \overline{K}(\text{Spec } C(X))\). The following are equivalent.

(i) \(O(A) \subseteq V\) (hence \(A \subseteq \text{int } V\) by (4.7)).

(ii) there are some \(\varepsilon > 0\) and some \(f \in C(X)\) with \(V = V(\{|f| \leq \varepsilon\})\) such that for every \(\delta > 0\) we have \(A \subseteq V(\{|f| \leq \delta\})\).

(iii) for every \(\varepsilon > 0\) there is some \(f \in C^*(X)\) with \(V = V(\{|f| \leq \varepsilon\})\) such that \(A \subseteq V(\{|f| = 0\})\).

**PROOF.** (iii)\(\Rightarrow\)(ii) is obvious.

(ii)\(\Rightarrow\)(i). If for some \(\varepsilon > 0\) and some \(f \in C(X)\) we have \(V = V(\{|f| \leq \varepsilon\})\) and \(A \subseteq V(\{|f| \leq \varepsilon\})\), then \(O(A) \subseteq V, A \subseteq V(\{|f| \leq \varepsilon\}) \subseteq D(\{|f| \geq \varepsilon\}) \subseteq V(\{|f| \leq \varepsilon\}) = V(\{|f| = 0\})\), so \(A\) is in the interior of \(V\).

(i)\(\Rightarrow\)(iii). If \(O(A) \subseteq V\), then \(A\) is in the interior of \(V\) and since \(\text{Spec } C(X)\) is completely normal there are \(W \in \overline{K}(\text{Spec } C(X)), U \in \overset{\circ}{K}(\text{Spec } C(X))\) with \(A \subseteq W \subseteq U \subseteq V\). Take \(f_0, g, h \in C(X)\) with \(V = V(\{|g| = 0\}), U = D(\{|h| = 0\})\) and \(W = V(\{|f_0| = 0\})\). Then \(W \subseteq U \subseteq V\) means \(\{f_0 = 0\} \subseteq \{h \neq 0\} \subseteq \{g = 0\}\) and by (4.24) there is some \(f \in C^*(X)\)
with \( \{ f = 0 \} = \{ f_0 = 0 \} \) and \( \{ f \leq \varepsilon \} = \{ g = 0 \} \). Hence \( V = V(\{ g = 0 \}) = V(\{|f| \leq \varepsilon\}) \) and \( A \subseteq V(\{ f_0 = 0 \}) = V(\{ f = 0 \}) \).

(4.31) Corollary. Let \( f \) be a filter of zero sets of \( X \) and let \( g := \{ Z \subseteq X \mid \text{there are some } f \in C^*(X) \text{ and some } \varepsilon \in \mathbb{R}, \varepsilon > 0 \text{ such that } Z = \{|f| \leq \varepsilon\} \text{ and such that } \{|f| \leq \delta\} \in f \text{ for all } \delta \geq 0 \} \).

Then \( g \) is the filter of zero sets of \( X \) with \( V(g) = \Theta(V(f)) \).

Proof. Let \( Z \) be a zero set of \( X \). Then by (4.30), \( Z \in g \) if and only if \( \Theta(V(f)) \subseteq V(Z) \). \( \square \)

(4.32) Definition. In \([\text{Gil-Jer}],\) \( 2L \), \( \varepsilon \)-ideals and \( \varepsilon \)-filters are defined as follows. An ideal \( a \) of \( C^*(X) \) is called an \( \varepsilon \)-ideal if \( a = \{ f \in C^*(X) \mid \forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \text{ there is some } g \in a \text{ and some } \delta > 0, \text{ such that } \{|f| \leq \varepsilon\} = \{|g| \leq \delta\} \} \).

A filter \( f \) of zero sets of \( X \) is called an \( \varepsilon \)-filter if \( f = \{|f| \leq \varepsilon\} \mid f \in C^*(X) \) and for all \( \delta > 0 \) we have \( \{|f| \leq \delta\} \in f \).

So by (4.31), a filter \( f \) of zero sets of \( X \) is an \( \varepsilon \)-filter if and only if \( \Theta(V(f)) = V(f) \).

In order to analyze the notion "\( \varepsilon \)-ideal" we need a preparation.

(4.33) Definition. An ideal \( a \) of \( C(X) \) is called an \( \varepsilon \)-ideal if
\[
\{ f \in C(X) \mid \forall \varepsilon \in C(X), \varepsilon > 0 \text{ everywhere } \exists g \in a : \{|f| \leq \varepsilon\} = \{|g| \leq \varepsilon\} \}.
\]

By (4.29), if \( X \) is compact, then \( a \) is an \( \varepsilon \)-ideal if and only if \( a = \{ f \in C(X) \mid \forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \text{ everywhere } \exists g \in a : \{|f| \leq \varepsilon\} = \{|g| \leq \varepsilon\} \} \). By applying (4.29) again for arbitrary \( X \), an ideal \( a \) of \( C^*(X) \) is an \( \varepsilon \)-ideal in the sense of \([\text{Gil-Jer}],\) \( 2L \) if and only if \( a \) is an \( \varepsilon \)-ideal of \( C(\beta X) \) in our sense. Hence both notions coincide, which justifies definition (4.33).

Observe that for an \( \varepsilon \)-ideal \( a \) of \( C(X) \) we have \( a = \{ f \in C(X) \mid \forall \varepsilon \in C(X), \varepsilon > 0 \text{ everywhere}, \text{ there is some } g \in O(a) \text{ with } \{|f| \leq \varepsilon\} = \{|g| \leq \varepsilon\} \} \). This follows from (4.27), which says that \( \{|f| \leq \varepsilon\} = \{|g| \leq \varepsilon\} \) guarantees some \( h \in O(g.C(X)) \subseteq O(a) \) with \( \{|f| \leq \varepsilon\} = \{|h| \leq 2\varepsilon|\} \).

(4.34) Lemma. \textit{Correct, with direct proof. Easier proof in (4.8).}

For all ideals \( a, b \) of \( C(X) \) we have \( O(a + b) = O(a) + O(b) \).

Proof. Since the \( O \)-operation is monotone we certainly have \( O(a + b) \supseteq O(a) + O(b) \). Conversely take \( h \in O(a + b) \). By (4.21), there are \( f \in a, g \in b \) and a cozero set \( D \) of \( X \) with
\[
\{ f + g = 0 \} \subseteq D \subseteq \{ h = 0 \}.
\]

By (4.24), there is some \( \rho \in C(X) \), \( \rho \geq 0 \) such that \( \{ f + g = 0 \} = \{ \rho = 0 \} \), \( D = \{ \rho < 3 \} \) and \( \{ h = 0 \} = \{ \rho \leq 3 \} \). Let \( F := -((1 - \rho \circ \frac{f^2}{1 + \rho^2}) \wedge 0) \) and \( G := -((1 - \rho \circ \frac{g^2}{1 + \rho^2}) \wedge 0) \) (since \( \{ f + g = 0 \} = \{ \rho = 0 \} \), these functions are well defined, cf. (3.4)).

Then \( \{ F = 0 \} \subseteq \{ \rho \circ \frac{f^2}{1 + \rho^2} < 1 \} \subseteq \{ \rho \circ \frac{g^2}{1 + \rho^2} \leq 1 \} = \{ F = 0 \} \). By (4.21), \( F \in O(f) \subseteq O(a) \). The same argument applied to \( g \) and \( G \) gives \( G \in O(b) \).

Finally \( \{ F + G = 0 \} = \{ \rho \circ \frac{f^2}{1 + \rho^2} \leq 1 \} \cap \{ \rho \circ \frac{g^2}{1 + \rho^2} \leq 1 \} \subseteq \{ \rho \circ \frac{f^2}{1 + \rho^2} + \rho \circ \frac{g^2}{1 + \rho^2} \leq 2 \} \subseteq \{ \rho < 3 \} \subseteq \{ h = 0 \} \). By (4.21) we get \( h \in O(F + G) \subseteq (F + G) \subseteq O(a) + O(b) \). \( \square \)

(4.35) Lemma and Definition. If \( a \) is an ideal of \( C(X) \), then
\[
e(a) := \{ f \in C(X) \mid O(f) \subseteq O(a) \}
\]
is a \( z \)-radical ideal of \( C(X) \) containing \( a \). If \( a \) is proper, then also \( e(a) \) is proper.
Moreover, \( f \in e(a) \) if and only if \( O(f) \subseteq a \) if and only if for every \( \varepsilon > 0 \) everywhere, there is some \( h \in a \) with \( \{|f| \leq \varepsilon\} = \{|h| \leq \varepsilon\} \); in particular \( a \) is an \( e \)-ideal if and only if \( e(a) = a \).

**Proof.** By (4.8), \( e(a) \) is an ideal of \( C(X) \). By (4.19.4), \( e(a) \) is \( z \)-radical. If \( a \) is proper, then \( O(1) = C(X) \not\subseteq a \); hence \( 1 \not\in e(a) \).

Since \( O(a) \subseteq a \) and \( O(f) \subseteq a \) implies \( O(f) = O(O(f)) \subseteq O(a) \) we have \( f \in e(A) \iff O(f) \subseteq a \) and it remains to show that \( f \in e(a) \) if and only if for every \( \varepsilon > 0 \) there is some \( h \in a \) with \( \{|f| \leq \varepsilon\} = \{|h| \leq \varepsilon\} \).

If \( \varepsilon \in C(X) \), \( \varepsilon > 0 \) everywhere and \( f \in C(X) \), then \( \{f = 0\} \subseteq \{|f| \leq \varepsilon\} \subseteq \{|f| < 2\varepsilon\} \subseteq \{|f| < 2\varepsilon\} \). By (4.27) there is some \( h \in O(f) \) such that \( \{|f| \leq \varepsilon\} = \{|h| \leq \varepsilon\} \). Hence, if \( f \in e(a) \) we get \( h \in a \) as desired.

Conversely take \( f \in C(X) \) such that for every \( \varepsilon \in C(X) \), \( \varepsilon > 0 \) everywhere there is some \( h \in a \) with \( \{|f| \leq \varepsilon\} = \{|h| \leq \varepsilon\} \);

Let \( g \in O(f) \). By (4.25) there is some \( \varepsilon \in C(X) \), \( \varepsilon > 0 \) everywhere, such that \( \{|f| \leq \varepsilon\} = \{|g = 0\} \). By assumption, there is some \( h \in a \) with \( \{|f| \leq \varepsilon\} = \{|h| \leq \varepsilon\} \). By (4.25) again, \( g \in O(a) \). This shows that \( O(f) \subseteq O(a) \), thus \( f \in e(a) \).

Observe that for \( f \in C(X) \) we have \( \sqrt{\mathbb{J}} = \bigcap V(f)^{\text{max}} \) hence \( \sqrt{\mathbb{J}} = \sqrt[\mathbb{R}]{\mathbb{J}} \). Together with (4.11) we get the following representation of \( \sqrt{\mathbb{J}} \):

\[
\sqrt{\mathbb{J}} = \{g \in C(X) \mid \forall h \in C(X) : h \in (h \cdot g) \Rightarrow h \in (h \cdot f)\}.
\]

**5. The \( \Upsilon \)-radical and \( a^\Upsilon \).**

(5.1) **Definition.** Let \( \Upsilon \) (the Greek letter “Upsilon”) denote the set of all continuous functions \( s : \mathbb{R} \to \mathbb{R} \) with \( \{s = 0\} = \{0\} \). Let \( a \) be an ideal of \( C(X) \). Then we define

\[
a^\Upsilon := \{f \in C(X) \mid s \circ f \in a \text{ for all } s \in \Upsilon\}.
\]

Here we consider \( \Upsilon \) as a set generalizing the power functions \( x^n \). Later we’ll consider \( \Upsilon \) also as a set generalizing the root functions \( \sqrt[n]{x} \), which fits well to the symbol “\( \Upsilon \)”.

We prove that \( a^\Upsilon \) is an ideal. First some observations.

(5.2) **Remark.** Let \( X \) be a Tychonoff space, let \( K \) be an ordered field and let \( f, g : X \to K \) be continuous maps. Let \( U \) be an open neighborhood of \( \{f = 0\} \) and let \( h : U \to K \) be continuous.

(i) If \( g = f \cdot h \) on \( U \), then the map \( q : X \to K \) defined by

\[
q(x) := \begin{cases} 
\frac{g(x)}{f(x)} & \text{if } f(x) \neq 0 \\
h(x) & \text{if } f(x) = 0
\end{cases}
\]

is continuous and \( g = f \cdot q \).

(ii) If \( |g| \leq |h| \cdot |f| \) on \( U \) and \( s : K \to K \) is a continuous map, such that \( \lim_{x \to 0} \frac{s(x)}{x} = 0 \) then the map \( q : X \to K \) defined by

\[
q(x) := \begin{cases} 
\frac{s(g(x))}{f(x)} & \text{if } f(x) \neq 0 \\
0 & \text{if } f(x) = 0
\end{cases}
\]
is continuous and \( s(g) = f \cdot q \).

(iii) If \( |g| \leq |h| \cdot |f| \) on \( U \) and \( f_1 : X \to K \) is a continuous map with \( \{ f = 0 \} \subseteq \{ f_1 = 0 \} \), then the map

\[
q(x) := \begin{cases} 
\frac{f_1(x)g(x)}{f(x)} & \text{if } f(x) \neq 0 \\
0 & \text{if } f(x) = 0
\end{cases}
\]

is continuous and \( f_1 \cdot g = f \cdot q \).

**Proof.** (i) is obvious.

(ii). If \( x \in U \) and \( g(x) \neq 0 \), then \( |q(x)| \leq \frac{|s(g(x))|}{|g(x)|} \cdot h(x) \). From the assumption on \( s \) we get for each \( a \in X \) with \( f(a) = 0 \), that \( \lim_{x \to a} q(x) = 0 = q(a) \).

(iii). The map

\[
f_2(x) := \begin{cases} 
\frac{g(x)}{f(x)} & \text{if } f(x) \neq 0 \\
0 & \text{if } f(x) = 0
\end{cases}
\]

is bounded in a neighborhood of each zero of \( f \). It follows that \( f_1 \cdot f_2 \) is continuous everywhere.

(5.3) **Corollary.** If \( f, g \in C(X) \) with \( \{ f = 0 \} \subseteq \int \{ g = 0 \} \) then \( g \in (f) \).\( ^0 \).

**Proof.** By (5.2)(i), we know that \( g \in (f) \) if \( \{ f = 0 \} \subseteq \int \{ g = 0 \} \). So if \( h \in C(X) \) with \( \{ g = 0 \} \subseteq \{ h = 0 \} \), then also \( h \in (f) \). This shows \( g \in (f) \).\( ^0 \).

(5.4) **Lemma.** Let \( f, g, h : \mathbb{R} \to \mathbb{R} \) be continuous with \( f(0) = 0 = g(0) \). Let \( X \) be a Tychonoff space, let \( F \in C(X), \varepsilon \in \mathbb{R}, \varepsilon > 0 \) and let \( |g| \leq |f \cdot h| \) on \( (-\varepsilon, \varepsilon) \). Then for all \( k \in \mathbb{N}, (g \circ F)^k \) is in the ideal generated by \( F^k \) and \( (f \circ F)^2 \) in \( C(X) \).

**Proof.** Let \( a \) be the ideal generated by \( F^k \) and \( (f \circ F)^2 \). In this proof, by a power \( s^n \) of a function \( s : \mathbb{R} \to \mathbb{R} \) we mean the map \( s^n(x) := s(x)^n \), not the \( n \)-fold iteration of \( s \).

By assumption \( h_1 := g^2 \vee (f \cdot h)^2 \) vanishes in \( (-\varepsilon, \varepsilon) \). Since \( F^{2^k} \cdot (h_1 \circ F) \in a \), we have \( \frac{1}{\varepsilon} F^{2^k} \cdot (h_1 \circ F) \in a \). From \( 0 \leq h_1 \circ F \leq \frac{1}{\varepsilon} F^{2^k} \cdot (h_1 \circ F) \) we get \( (h_1 \circ F)^2 \in a \) (since \( C(X) \) is real closed). By assumption \( (g^2 \vee (f \cdot h)^2)^2 \circ F = \frac{1}{\varepsilon} F^{2^k} \cdot (h_1 \circ F)^2 \in a \) (since \( 0 \leq g^4 \leq (g^2 \vee (f \cdot h)^2)^2 \) we get \( (g \circ F)^k \in a \) from (5.2)(ii).

(5.5) **Definition.** A subset \( \Upsilon_0 \) of \( \Upsilon \) is called a set of generalized root functions if for all \( s \in \Upsilon \), there is some \( s_0 \in \Upsilon_0 \) and some \( \varepsilon \in \mathbb{R}, \varepsilon > 0 \) with \( |s| \leq |s_0| \) on \( (0, \varepsilon) \).

A subset \( \Upsilon_0 \) of \( \Upsilon \) is called a set of generalized power functions if for all \( s \in \Upsilon \), there is some \( s_0 \in \Upsilon_0 \) and some \( \varepsilon \in \mathbb{R}, \varepsilon > 0 \) with \( |s_0| \leq |s| \) on \( (0, \varepsilon) \).

For example the set of all homeomorphisms \( \mathbb{R} \to \mathbb{R} \) mapping 0 to 0 is a set of generalized root functions and a set of generalized power functions:

(5.6) **Lemma.** Let \( a, b : (0, \infty) \to \mathbb{R} \) be functions. Suppose \( \lim_{t \to 0^+} b(t) = 0 \) for all \( t \) and \( a \) is increasing on some interval \( (0, \varepsilon) \).

Then there are an increasing homeomorphism \( s \in \Upsilon \) and some \( \delta > 0 \) with \( s(a(t)) > b(t) \) for all \( t \in (0, \delta) \).

**Proof.** By replacing \( a(t) \) with \( t \cdot a(t) \) we may assume that \( a(t) \) is strictly increasing in some interval \( I \) with endpoint 0. By replacing \( b(t) \) with \( \sup \left\{ b(t') \mid 0 < t' \leq t \right\} \) we may also assume that \( b(t) \) is increasing in \( I \). Since \( \lim_{t \to 0^+} b(t) = 0 \), it is easy to construct an increasing homeomorphism \( s_1 : \Upsilon \to \Upsilon \) with \( s_1(t) > b(t) \) for all \( t \in I \). Similarly, since \( a > 0 \) it is easy to construct an increasing homeomorphism \( s_2 : \Upsilon \to \Upsilon \) with \( s_2(t) < a(t) \) for all \( t \in I \). Then \( s := s_1 \circ s_2^{-1} \in \Upsilon \) is an increasing homeomorphism and for \( t \in I \) we have \( s_2^{-1}(a(t)) > t \), so \( s(a(t)) = s_1 \circ s_2^{-1}(a(t)) > s_1(t) > b(t) \).\( ^0 \)
Computation of the $z$-radical in $C(X)$

(5.7) **Proposition.** If $a$ is an ideal of $C(X)$, then $a^T$ is an ideal and the largest subset of $a$ that is closed under composition with all $s \in \Upsilon$. If $\Upsilon_0 \subseteq \Upsilon$ is a set of generalized root functions, then

$$a^T = \{F \in a \mid s_0 \circ F, s_0 \circ (-F) \in a \text{ for all } s \in \Upsilon_0\}$$

**Proof.** By definition, $a^T$ is the largest subset of $a$ that is closed under composition with all $s \in \Upsilon$. We have to show that $a^T$ is an ideal.

Claim 1. $a^T = \{F \in a \mid s_0 \circ F, s_0 \circ (-F) \in a \text{ for all } s \in \Upsilon_0\}$.

Let $F \in a$ such that $s_0 \circ F, s_0 \circ (-F) \in a$ for all $s \in \Upsilon_0$ and let $s \in \Upsilon$. Choose $s_0, s_1 \in \Upsilon_0$ and some $\varepsilon > 0$ such that $|s(u)| \leq s_0(u)^2$ and $|s(u)| \leq s_1(u)^2$ for $u \in (0, \varepsilon)$. Then $s_2 \circ F \in a$ and $s_2(u) := s_0(u)^2 + s_1(-u)^2$ and $|s(tu)| \leq s_2(u)$ for $u \in (-\varepsilon, \varepsilon)$. By (5.4) we have $(s \circ F) \in a$. If we replace $s$ by $\sqrt{s}$ we get $s \circ F \in a$.

For the rest of the proof we work with the set $\Upsilon_0$ of all $s \in \Upsilon$, which are increasing, $> 0$ on $(0, +\infty)$ and symmetric, i.e. $s(-u) = s(u)$. By (5.6), $\Upsilon_0$ is a set of generalized root functions.

Claim 2. If $F \in a^T$ and $G \in C(X)$ with $|G| \leq |F|$, then $G \in a^T$.

We have $G^3 = F \cdot H$ for some $H \in C(X)$ since $C(X)$ is real closed. Thus $G = \sqrt{F \cdot H} = \sqrt{F} \cdot \sqrt{H} \in a$. If $s \in \Upsilon_0$, then $s$ is symmetric and increasing, hence $0 \leq s \circ G \leq s \circ F$. Since $s \circ F \in a^T$, we get $s \circ G \in a$ as we have just proved for $G$. By Claim 1 we have $G \in a^T$.

Claim 3. $a^T$ is a subgroup of $C(X)$.

If $F, G \in a^T$, then $|F| - |G| \leq |F| + |G| \leq 2(|F| \vee |G|)$ and by Claim 2 it remains to show that $|F| \vee |G| \in a^T$ for all $F, G \in a^T$. By definition we have $|F|, |G| \in a^T$. Take $s \in \Upsilon_0$. Then $s \circ (|F| \vee |G|) \leq s \circ |F| + s \circ |G| \in a$. Thus $(s \circ (|F| \vee |G|))^2 \in a$ for all $s \in \Upsilon_0$ and consequently $s \circ (|F| \vee |G|) \in a$ for all $s \in \Upsilon_0$.

Claim 4. If $F \in a^T$, $H \in C(X)$ with $F \cdot H \geq 0$ and $s_0 \in \Upsilon_0$, then $(s_0 \circ (F \cdot H))^2 \in a$.

We have $s_0 \circ (F \cdot H) \leq s_0 \circ (F - (1 + H)) \leq s_0 \circ \sqrt{F}$ on $\{\sqrt{F} < \frac{1}{1+H}\}$, since $s_0$ is increasing on $(0, +\infty)$. Since $s_0(u) \neq 0$ for $u \neq 0$, it follows that the zeros of $s_0 \circ \sqrt{F}$ are in the interior of $\{s_0 \circ (F \cdot H) \leq s_0 \circ \sqrt{F}\}$, and hence $(s_0 \circ (F \cdot H))^2 \in a$.

In order to prove that $a^T$ is an ideal, it remains to show that $F \cdot H \in a^T$ for all $F \in a^T$ and all $H \in C(X)$. By Claim 2 we may assume that $F \cdot H \geq 0$. Since $\{s_0 \circ s_0 \mid s_0 \in \Upsilon_0\}$ also has the properties assumed for $\Upsilon_0$, we get $F \cdot H \in a^T$ from Claim 4 and Claim 1.

(5.8) **Definition.** An ideal $a$ of $C(X)$ is called $\Upsilon$-radical if $s \circ f \in a$ for all $f \in a$ and all $s \in \Upsilon$. Since the intersection of $\Upsilon$-radical ideals is obviously again $\Upsilon$-radical, we may define the $\Upsilon$-radical of an ideal $a$ of $C(X)$ as the smallest $\Upsilon$-radical ideal of $C(X)$ containing $a$. We write $\sqrt{a}$ for the $\Upsilon$-radical of $a$.

Clearly $\Upsilon$-radical ideals are ideals and every $z$-radical ideal of $C(X)$ is $\Upsilon$-radical.

(5.9) **Corollary.** Let $a$ be an ideal of $C(X)$. Then

(i) $a^T$ is the largest $\Upsilon$-radical ideal contained in $a$, in particular $a^0 \subseteq a^T$.

(ii) If $b$ is another ideal of $C(X)$, then

$$\sqrt{a + b} = \sqrt{a} + \sqrt{b}.$$ 

(iii) If $a$ is $\Upsilon$-radical then every $p \in V(a)^{\min}$ is $\Upsilon$-radical, too.

**Proof.** (i) follows from (5.7).

(ii). $(\sqrt{a} + \sqrt{b})^T$ contains $\sqrt{a}$ and $\sqrt{b}$, hence by (i) it is equal to $\sqrt{a} + \sqrt{b}$.

(iii). If $p \in V(a)^{\min}$ and $a$ is $\Upsilon$-radical, then $a \subseteq p^T$. Since $p^T$ contains every minimal prime ideal of $C(X)$ that specializes to $p$, $p^T$ is prime as well, which implies $p^T = p$.  \[\square\]
Let $X$ be a Hausdorff space and let $f, g : X \to \mathbb{R}$ be continuous. If \{\(g(x) = 0\}\} \subseteq \text{int}\{f = 0\}$, then the function
\[
h : x \to \begin{cases} \frac{f(x)}{g(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}
\]
is well defined and continuous with $f = h \cdot g$. If $f$ has compact support, then $|h| \leq c \cdot |f|$ for some $c \in \mathbb{R}$.

**Proof.** Since \{\(f \neq 0\}\} \subseteq \{\(g \neq 0\}\}, $h$ is well defined and $h$ is continuous on \{\(f \neq 0\}\}. If $x \in \text{int}\{f = 0\}$, then $h$ clearly is continuous in $x$. If $f(x) = 0$ and $x \notin \text{int}\{f = 0\}$, then by assumption $g(x) \neq 0$ and $|g(x)| \geq \varepsilon > 0$, in some neighborhood of $x$, for some $\varepsilon > 0$. But then $\lim_{\to x} y g(y) = 0$, as $f(x) = 0$. Clearly $f = h \cdot g$.

Now suppose $K := \{f \neq 0\} \subseteq X$ is compact. Since \{\(g = 0\}\} \subseteq \text{int}\{f = 0\}$, $|g|$ has a minimum value $\varepsilon > 0$ on $K$. Thus $c := \frac{1}{\varepsilon}$ satisfies $|h| \leq c \cdot |f|$.

**Corollary.** Recall from [Pr-Schm] that a reflexive and transitive binary relation $\preceq$ on a ring $A$ is a radical relation if for all $a, b, c \in A$ we have $a \preceq a^2, a \preceq 1, 1 \preceq 0, a \preceq b \Rightarrow ac \preceq bc$ and $a, b \preceq c \Rightarrow a + b \preceq c$.

(i) The binary relation $f \preceq g : \Leftrightarrow f \in \sqrt{(g)}$ is a radical relation of the ring $C(X)$.

(ii) The binary relation $f \preceq g : \Leftrightarrow f \in \sqrt{(g)}$ is a radical relation of the ring $C(X)$.

**Proof.** (i) follows immediately from the definition.

(ii). Easily, $\preceq$ is reflexive, transitive, $1 \not\preceq 0$ and for all $f \in C(X)$ we have $f \preceq f^2, f \preceq 1$. Let $f, g, h \in C(X)$ with $f \preceq g$, hence $f \in \sqrt{(g)}$. Then for every $\mathfrak{T}$-radical prime ideal $\mathfrak{p}$ of $\text{Spec}C(X)$ containing $g-h$ we have $f \cdot h \in \mathfrak{p}$. So by (5.9), $f \cdot h \in \sqrt{(g-h)}$, thus $f \cdot h \preceq g-h$.

Finally, if also $h \preceq g$, then clearly $f + h \preceq g$.

**Corollary.** Let $X$ be a Tychonoff space so that every $\mathfrak{T}$-radical prime ideal of $C(X)$ is $z$-radical. Then every $\mathfrak{T}$-radical ideal of $C(X)$ is $z$-radical.

**Proof.** Let $a$ be $\mathfrak{T}$-radical. By (5.9), we know that $a$ is the intersection of $\mathfrak{T}$-radical prime ideals, hence by assumption $a^2$ is $z$-radical.

The ideals $\sqrt{a}$ are important, since they are computable in the following sense:

**Proposition.** Let $a$ be an ideal of $C(X)$. Then

(i) For every set $\mathfrak{T}_0 \subseteq \mathfrak{T}$ of generalized root functions we have
\[
\sqrt{a} = \{g \cdot (s \circ f) \mid g \in C(X), f \in a, f \geq 0, s \in \mathfrak{T}_0\}.
\]

(ii) For every set $\mathfrak{T}_0 \subseteq \mathfrak{T}$ of generalized power functions we have
\[
\sqrt{a} = \{f \in C(X) \mid s \cdot |f| \in a \text{ for some } s \in \mathfrak{T}_0\}.
\]

**Proof.** (i). Let $b$ be the ideal generated by all the $s \circ f$ with $f \in a, s \in \mathfrak{T}$. By definition of $b^\mathfrak{T}$ we have $a \subseteq b^\mathfrak{T}$. Since $b^\mathfrak{T}$ is an $\mathfrak{T}$-radical ideal we have $\sqrt{a} = b^\mathfrak{T}$. Therefore we know $\sqrt{a} = \{g \cdot (s \circ f) \mid g \in C(X), f \in a, s \in \mathfrak{T}\}$ if we show that $b$ is contained in the set on the right hand side.
To see this, take \( h = g_1(s_1 \circ f_1) + \ldots + g_n(s_n \circ f_n) \in b \) with \( g_1, \ldots, g_n \in C(X), s_1, \ldots, s_n \in \mathcal{T} \) and \( f_1, \ldots, f_n \in a \). Take \( s \in \mathcal{T} \) such that \( |s_i(u)| \leq s(u_2^B) \) \((1 \leq i \leq n)\) and such that \( s \) is increasing on \((0, \infty)\). Let \( g := |g_1| \cup \ldots \cup |g_n| \) and let \( f = f_1^2 + \ldots + f_n^2 \in a \). Then \( |h| \leq n \rho(s \circ f), f \geq 0 \) and there is some \( g^* \in C(X) \) with \( h^3 = g^* \circ (s \circ f) \). Therefore \( h = \sqrt[3]{g^* \circ (s \circ f)} \).

Thus we know \( \sqrt[3]{a} = (g \circ (s \circ f)) | g \in C(X), f \in a, f \geq 0, s \in \mathcal{T} \) and (i) holds if we can show that each \( g \circ (s \circ f) \) from \( \sqrt[3]{a} \) is of the form \( g^* \circ (s \circ f^*) \) for some \( g^* \in C(X), f^* \in a, f^* \geq 0 \) and \( s \in \mathcal{T}_0 \).

Take \( s_0 \in \mathcal{T}_0 \) such that \( |s| \leq |s_0| \) on some \((0, \varepsilon)\). Then \( |g \circ (s \circ f)| \leq |g(\rho) \circ (s \circ f)| \) on \( \{f \leq \varepsilon\} \). Since \( g \neq 0 \) and \( s_0 \) does not vanish in \((0, \infty)\), the zeros of \( s_0 \circ f \) are in \( \{f \leq \varepsilon\} \). So by (5.2)(ii), \( (g \circ (s \circ f))^3 = g_1 \circ (s \circ f)^3 \) for some \( g_1 \in C(X) \) and finally \( g \circ (s \circ f) = g^* \circ (s \circ f^*) \) with \( g^* = \sqrt[3]{g^*} \) and \( f^* = f \geq 0 \).

(ii). First let \( f \in C(X) \) and \( s \in \mathcal{T} \) with \( s(f) \in a \). Take an homeomorphism \( t \in \mathcal{T} \) such that \( |t| \leq |s^2| \) on some \((-\varepsilon, \varepsilon)\). Then \( |t \circ f| \leq |(s \circ f^2)| \) on \( \{|f| < \varepsilon\} \), hence by (5.2)(ii), \( |t \circ f| = g \circ (s \circ f)^3 \) for some \( g \in C(X) \). Consequently, \( t \circ f = \sqrt[3]{g \circ (s \circ f)} \in a \) and \( f = t^{-1} \circ t \circ f \in \sqrt[3]{a} \).

Conversely let \( h \in \sqrt[3]{a} \). By (i) applied to the set of root functions
\[
\{s(u^3) \mid s \in \mathcal{T}, s \text{ an increasing homeomorphism}\},
\]there is an increasing homeomorphism \( s \in \mathcal{T} \) with \( h = g \circ (s \circ f^2) \) for some \( g \in C(X) \) and some \( f \in a, f \geq 0 \). Observe that \( s \circ f^2 \geq 0 \). Then \( |h| \leq 1 + |s| \circ (s \circ f^2) \leq |s \circ f^2| \) on \( \{s \circ f^2 \leq 1 \} \). Let \( \varepsilon > 0 \) and let \( t \in \mathcal{T}_0 \) such that \( |t(u)| \leq s^{-1}(u^2) \) for \( u \in (0, \varepsilon) \).

Then \( t \circ |h| \leq s^{-1} \circ |h|^2 \) on \( U := \{s \circ f^3 \leq 1 \} \). Since \( s \) is increasing, \( s^{-1} \) is increasing and \( |h| \leq s \circ f^3 \) on \( U \) implies \( s^{-1} \circ |h|^2 \leq s \circ f^2 \) on \( U \). This shows that \( t \circ |h| \leq f^3 \) on \( U \). Since \( U \) is an open neighborhood of \( \{f = 0\} \), (5.2)(ii) gives \( (t \circ |h|^2)^3 = g_1 \circ f^3 \) for some \( g_1 \in C(X) \), thus \( t \circ |h| = \sqrt[3]{g^*} \circ f \) as desired. \( \square \)

(5.14) COROLLARY AND DEFINITION. The set \( \sqrt[3]{\text{Spec}C(X)} \) of \( \mathcal{T} \)-radical prime ideals is a proconstructible subset of \( \text{Spec}C(X) \). If \( \tau : X \to Y \) is a continuous map between Tychonoff spaces \( X, Y \) and \( \varphi : C(Y) \to C(X) \) is the corresponding ring homomorphism - so \( \varphi(g) = g \circ \tau \) - then for every ideal \( a \) of \( C(X) \), we have
\[
\sqrt[3]{\varphi^{-1}(a)} = \varphi^{-1}(\sqrt[3]{a}) \text{ and } \varphi^{-1}(a)^\mathcal{T} = \varphi^{-1}(a)^Y.
\]
In particular, \( \text{Spec} \varphi \) maps the \( \mathcal{T} \)-radical prime ideals of \( C(X) \) to the \( \mathcal{T} \)-radical prime ideals of \( C(Y) \).

PROOF. A prime ideal \( p \) of \( C(X) \) is in \( \sqrt[3]{\text{Spec}C(X)} \) if and only if for all \( f \in C(X) \) and each \( s \in \mathcal{T} \) we have \( f \not\in p \) or \( s \circ f \in p \). In other words
\[
\sqrt[3]{\text{Spec}C(X)} = \bigcap_{f \in C(X), s \in \mathcal{T}} D(f) \cup V(s \circ f).
\]
Hence \( \sqrt[3]{\text{Spec}C(X)} \) is a proconstructible subset of \( \text{Spec}C(X) \). We use (5.13)(ii) for the computation of the \( \mathcal{T} \)-radicals (with \( \mathcal{T}_0 = \mathcal{T} \)).

Let \( g \in C(Y) \). Then \( g \in \sqrt[3]{\varphi^{-1}(a)} \) if and only if \( g \circ (s \circ f) \) for some \( s \in \mathcal{T} \). Since \( \varphi(s \circ g) = s \circ g \circ \tau = s \circ \varphi(g) \) we get \( g \in \sqrt[3]{\varphi^{-1}(a)} \) if and only if \( g \circ \varphi(g) \in a \) for some \( s \in \mathcal{T} \) if and only if \( g \in \sqrt[3]{a} \). This shows \( \sqrt[3]{\varphi^{-1}(a)} = \varphi^{-1}(\sqrt[3]{a}) \).

Finally \( g \in \varphi^{-1}(a)^\mathcal{T} \) if and only if \( g \circ s \circ f \) for all \( s \in \mathcal{T} \) if and only if \( g \circ \varphi(g) \in a \) for all \( s \in \mathcal{T} \) iff \( g \in \varphi^{-1}(a)^Y \). \( \square \)
convexity does not hold in general, cf. (8.5) and (8.18) below. The convexity of \( \text{Spec} \ C \) remains valid for \( \Upsilon \)-radical prime ideals. For prime \( \varphi \)-ideals, the convexity does not hold in general, cf. (8.5) and (8.18) below.

Note that a convex map does not map \( A \rightarrow \mathbb{R} \) increasing. We call this set the convex hull of \( Z \).

6. Convexity of \( \Upsilon \)-Spec

Let \( \tau : X \rightarrow Y \) be a continuous map between Tychonoff spaces \( X, Y \) and let \( \varphi : C(Y) \rightarrow C(X) \) be the corresponding ring homomorphism. It is well known that \( \text{Spec} \varphi \) is a convex map. A crucial advantage of the \( \Upsilon \)-radical prime ideals over the prime \( \varphi \)-ideals is that the convexity of \( \text{Spec} \varphi \) remains valid for \( \Upsilon \)-radical prime ideals. For prime \( \varphi \)-ideals, the convexity does not hold in general, cf. (8.5) and (8.18) below.

First we recall some folklore facts about convex sets and maps in the category of spectral spaces. We omit the easy proofs.

A subset \( Z \) of a spectral space \( X \) is called a convex set if \( z_1 \prec x \prec z_2 \) with \( z_1, z_2 \in Z \) and \( x \in X \) implies \( x \in Z \). A map \( f : X \rightarrow Y \) between spectral spaces \( X, Y \) is called a convex map if for all \( x_1, x_2 \in X \) the set \( \{ f(x) \mid x \in X, \ x_1 \prec x \prec x_2 \} \) is a convex subset of \( Y \).

Note that a convex map does not map \( X \) onto a convex subset of \( Y \) in general! For example if \( X \) is boolean, then \( f \) is convex, but in general \( f(X) \) is not convex in \( Y \) (the inclusion \( (z-\text{Spec} \ C(\mathbb{R}^n))^{\text{con}} \rightarrow \text{Spec} C(\mathbb{R}^n) \) is a nice counter example in our context).

Convex subsets of spectral spaces can be described in the following way:

(6.1) Proposition. (Convex hull formula)

Let \( Z \) be a quasi-compact subset of a spectral space \( X \), which is also quasi-compact in the inverse topology of \( X \). Then

\[
\{ x \in X \mid \exists z_1, z_2 \in Z : z_1 \prec x \prec z_2 \} = \bigcap_{A \in \mathbb{R}(X), \ O \in \mathbb{X}(X), Z \subseteq A \cap O} A \cap O = \overline{Z} \cap \overline{Z}^{pp}.
\]

We call this set the convex hull of \( Z \).
Computation of the $z$-radical in $C(X)$

\[ \text{LONG VERSION} \]

**Proof.** The equality $\bigcap_{A \in \mathcal{K}(X), \ O \in \mathcal{K}(X), \ z \subseteq A \cap O} A \cap O = Z \cap Z^{\text{pp}}$ holds, since $Z(X) = \bigcap_{A \in \mathcal{K}(X), \ z \subseteq A} A$ and $Z^{\text{pp}} = \bigcap_{O \in \mathcal{K}(X), \ z \subseteq O} O$.

Since $Z$ is quasi-compact in the inverse topology, $Z = \{ x \in X \mid \exists z \in Z : z \leadsto x \}$. Since $Z$ is quasi-compact, $Z^{\text{pp}} = \{ x \in X \mid \exists z \in Z : x \leadsto z \}$. Hence $Z \cap Z^{\text{pp}}$ is indeed equal to the set on the left hand side of the proposition. \hfill \Box

\[ \text{END OF LONG VERSION} \]

In particular, the convex hull of $Z$ is proconstructible if $Z$ is quasi-compact in $X$ and in $X^{\text{pp}}$.

(6.2) Corollary. Let $Z$ be a quasi-compact subset of a spectral space $X$, which is also quasi-compact in the inverse topology of $X$. Then

(i) $Z$ is convex if and only if $Z$ is the intersection of constructible, convex sets (in particular, $Z$ is proconstructible).

(ii) $Z$ is constructible and convex if and only if $Z = A \cap O$ for some $A \in \mathcal{K}(X)$ and some $O \in \mathcal{K}(X)$.

**Proof.** (i) follows from (6.1) and (ii) is implied by (i) and compactness of $X^{\text{con}}$. \hfill \Box

Another consequence of the convex hull formula (6.1) is:

(6.3) Lemma. Let $f : X \longrightarrow Y$ be a map between spectral spaces $X, Y$. The following are equivalent.

(i) $f$ maps convex constructible subsets of $X$ onto convex subsets of $Y$.

(ii) $f$ maps convex proconstructible subsets of $X$ onto convex subsets of $Y$.

If this is the case, then $f$ is convex. \hfill \Box

\[ \text{LONG VERSION} \]

**Proof.** We only have to show (i) $\Rightarrow$ (ii). Then (ii) implies the convexity of $f$, which means by definition, that the image of the convex proconstructible set $\{ x_1 \} \cap \text{Gen} x_2$ is convex.

So assume (i) and let $Z \subseteq X$ be proconstructible and convex. By (6.1), $Z$ is the intersection of all sets $A \cap O$, where $A \in \mathcal{K}(X)$, $O \in \mathcal{K}(X)$ and $Z \subseteq A \cap O$. Since these sets are quasicompact and the collection of these sets is closed under finite intersections, we have $f(Z) = \bigcap_{A \in \mathcal{K}(X), \ O \in \mathcal{K}(X), \ z \subseteq A \cap O} f(A \cap O)$.

By (i), the sets $f(A \cap O)$ are convex, hence $f(Z)$ is convex, too. \hfill \Box

Direct proof: Pick $y \in Y$, with $f(z_1) \leadsto y \leadsto f(z_2)$ for some $z_1, z_2 \in Z$. We have to show that $f^{-1}(y) \cap Z \neq \emptyset$. Since $f^{-1}(y)$ and $Z$ are proconstructible subsets of $X$, (6.1) and the convexity of $Z$ implies that we only have to show $f^{-1}(y) \cap A \cap O \neq \emptyset$ for all $A \in \mathcal{K}(X)$, $O \in \mathcal{K}(X)$ with $Z \subseteq A \cap O$.

By assumption $f(A \cap O)$ is convex. Since $f(z_1), f(z_2) \in f(A \cap O)$ it follows $y \in f(A \cap O)$, hence indeed $f^{-1}(y) \cap A \cap O \neq \emptyset$. \hfill \Box

\[ \text{END OF LONG VERSION} \]

Next we recall a remarkable property of the spectral map induced by a ring homomorphism on the real spectra of the rings.

(6.4) Fact. Let $\varphi : A \longrightarrow B$ be a ring homomorphism between arbitrary rings. Then the map $\text{Sper} \varphi : \text{Sper} B \longrightarrow \text{Sper} A$ is convex.
Proof. This is well known; a proof can be found in [Kn-Sch], III.7. Korollar 4. □
Hence, if $A$ and $B$ are real closed rings, then $\text{Spec } \varphi$ is convex - this applies to our situation with $A = C(Y)$ and $B = C(X)$.

(6.5) THEOREM. Let $\tau : X \to Y$ be a continuous map between Tychonoff spaces $X$, $Y$ and let $\varphi : C(Y) \to C(X)$ be the corresponding ring homomorphism, so $\varphi(g) = g \circ \tau$. Then the restriction $\tau^* \text{Spec } \varphi$ of $\text{Spec } \varphi$ to $\tau^* \text{Spec } C(X)$ is a convex map $\tau^* \text{Spec } C(X) \to \tau^* \text{Spec } C(Y)$.

Proof. Let $p, q \in \tau^* \text{Spec } C(X)$, $p \subseteq q$ and let $r \in \tau^* \text{Spec } C(Y)$ with $\varphi^{-1}(p) \subseteq r \subseteq \varphi^{-1}(q)$. We have to find some $r_0 \in \tau^* \text{Spec } C(X)$ with $r = \varphi^{-1}(r_0)$ and $p \subseteq r_0 \subseteq q$. By (6.4), there is some $r^* \in \text{Spec } C(X)$ with $r = \varphi^{-1}(r^*)$ and $p \subseteq r^* \subseteq q$. Now we apply (5.14). Then $r = \sqrt{r}^* = \sqrt{\varphi^{-1}(r)} = \varphi^{-1}(\sqrt{r^*})$ and we can take $r_0 := \sqrt{r^*}$. This shows that $\tau^* \text{Spec } \varphi$ is a convex map. □

Recall from commutative algebra, that a spectral map $f : S \to S'$ between spectral spaces has going-up if for all $x', y' \in S'$ and each $x \in S$ with $f(x) = x'$ there is some $y \in S$ with $x \sim y$ and $f(y) = y'$. The map $f$ has going-down if for all $x', y' \in S'$ and each $y \in S$ with $x' \sim y' = f(y)$ there is some $x \in S$ with $x \sim y$ and $f(x) = x'$.

(6.6) PROPOSITION. In the situation of (6.5), $\text{Spec } \varphi$ has going-up if and only if $\tau^* \text{Spec } \varphi$ has going-up and $\text{Spec } \varphi$ has going-down if and only if $\tau^* \text{Spec } \varphi$ has going-down.

Proof. We prove that $\text{Spec } \varphi$ has going down if $\tau^* \text{Spec } \varphi$ has going-down. The other implications are similar (and easier). Suppose $\tau^* \text{Spec } \varphi$ has going-down. Let $p', q' \in \text{Spec } C(Y)$ and $q \in \text{Spec } C(X)$ such that $p' \sim q' = \varphi^{-1}(q)$. By (5.14),

$$p'^* \sim q'^* = \varphi^{-1}(q^*) \sim q',$$

in particular $p'$ and $q'^*$ are comparable. If $q'^* \sim p'$, then the convexity of $\text{Spec } \varphi$ gives some $p \in \text{Spec } C(X)$ with $\varphi^{-1}(p) = p'$ and $q^* \subseteq p \subseteq q$. Therefore we may assume that $p' \sim q'^*$. Since $\tau^* \text{Spec } \varphi$ has going-down, there is some $p_* \in \tau^* \text{Spec } C(X)$ with $\varphi^{-1}(p_*) = p'^*$ and $p_* \subseteq q^*$. Now the convexity of $\text{Spec } \varphi$ gives some $p \in \text{Spec } C(X)$ with $p_* \subseteq p \subseteq q^*$ and $\varphi^{-1}(p) = p'$. This shows that $\text{Spec } \varphi$ has going-down. □

(6.7) COROLLARY. Let $\tau : X \to Y$ be a continuous map between Tychonoff spaces $X$, $Y$ and let $\varphi : C(Y) \to C(X)$ be the corresponding ring homomorphism. Then $\text{Spec } \varphi$ is surjective if and only if $\tau^* \text{Spec } \varphi$ is surjective.

Proof. First assume that $\text{Spec } \varphi$ is surjective. Let $q \in \tau^* \text{Spec } C(Y)$. By assumption there is some $p \in \text{Spec } C(X)$ with $q = \varphi^{-1}(p)$. So by (5.14), $\sqrt{q}$ is a preimage of $q$ under $\tau^* \text{Spec } \varphi$.

Now assume that $\tau^* \text{Spec } \varphi$ is surjective. Let $q \in \text{Spec } C(Y)$ and let $p \in \tau^* \text{Spec } C(X)$ with $\varphi^{-1}(p) = q^*$. UNDER CONSTRUCTION: Umkehrung fraglich □

7. Barrier functions
Certain functions from \( \Upsilon \) play an important role in the algebra of the rings \( C(X) \), namely those which form a “barrier” for polynomial root functions:

(7.1) **DEFINITION.** A **barrier function** is a function \( L : \mathbb{R} \rightarrow \mathbb{R} \) such that

(B1) \( L \) is continuous, increasing in \([0, +\infty)\) and \( 0 \leq L(x) = L(-x) \) for all \( x \in \mathbb{R} \).

(B2) \( \{L = 0\} = \{0\} \), hence 0 is the unique zero of \( L \).

(B3) (Barrier condition). For all \( n \in \mathbb{N} \) there is some \( \varepsilon > 0 \) such that \( \sqrt[n]{x} < L(x) \) for all \( x \in \mathbb{R} \) with \( 0 < x < \varepsilon \).

Observe that for any barrier function \( L \) and any semialgebraic map \( s : \mathbb{R} \rightarrow \mathbb{R} \) with \( \lim_{t \to 0} s(t) = 0 \) there is some \( \varepsilon > 0 \) such that \( s(t) < L(x) \) for all \( x \in \mathbb{R} \) with \( 0 < x < \varepsilon \).

For example the function

\[
L : \mathbb{R} \rightarrow \mathbb{R}
\]

\[
x \mapsto \begin{cases} \frac{1}{2n|x|} & \text{if } 0 < |x| < e^{-1} \\ 0 & \text{if } x = 0 \\ 1 & \text{if } |x| \geq e^{-1} \end{cases}
\]

is a barrier function. To see this we show that \( \sqrt[n]{x} < L(x) \) for all \( x \in \mathbb{R} \), \( 0 < x < n^{-2n} \).

Otherwise, \( \sqrt[n]{x} > L(x) \) implies \( \frac{1}{2n} \geq \log x \), hence \( e^{-\frac{1}{2n}} > x \) and \( e^\frac{1}{2} \leq \frac{1}{x} \) for all \( 0 < x < n^{-2n} \). But \( x < n^{-2n} \) means \( \sqrt[n]{x} < n^{-2} \), thus \( \frac{1}{\sqrt[n]{x}} > n^2 \) and this contradicts the exponential growth axiom “\( t > n^2 \Rightarrow e^t > t^n \)”.

Another barrier function, which pops up in [Gil-Jer] at several places is

\[
f(x) := \sum_{n=1}^{\infty} \frac{1}{2n} \sqrt[n-1]{|x|} \land 1
\]

(7.2) **LEMMA.** If \( L \) is a barrier function, \( p \in \text{Spec} \ C(X) \) and \( f \in C(X) \) with \( 1 \not\in p + (f) \), then \( \sqrt[n]{|f|} \mod p \leq L \circ f \mod p \) for all \( n \in \mathbb{N} \).

If in addition \( f \not\in p \), then \( \sqrt[n]{|f|} \mod p < L \circ f \mod p \) for all \( n \in \mathbb{N} \).

**PROOF.** Take \( \varepsilon > 0 \) such that \( \sqrt[n]{x} < L(x) \) for all \( x \in \mathbb{R} \) with \( 0 < x < \varepsilon \). Then \( \{|f| = 0\} \subseteq \{|f| < \varepsilon\} \subseteq \{\sqrt[n]{|f|} \leq L \circ f\} \) and from (4.23)(i) we get \( \sqrt[n]{|f|} \mod p \leq L \circ f \mod p \).

Now suppose \( \sqrt[n]{|f|} \mod p = L \circ f \mod p \) for some \( n \in \mathbb{N} \). Then \( \sqrt[n]{|f|} \mod p \leq L \circ f \mod p \). Since \( 1 \not\in p + (f) \) we have \( \sqrt[n]{|f|} \mod p \leq 2^n \sqrt[n]{|f|} \mod p \), hence \( \sqrt[n]{|f|} \equiv \sqrt[n]{|f|} \mod p \). So \( \sqrt[n]{|f|} \in p \) or \( 1 - \sqrt[n]{|f|} \in p \), which again means \( f \in p \) or \( 1 \in p + \sqrt[n]{|f|} \) in contradiction to our assumptions. \( \square \)

8. **COMPUTATION OF \( z \)-RADICALS**

First we do the case of compact spaces.

(8.1) **LEMMA.** Let \( X \) be a compact space. Then each \( \Upsilon \)-radical ideal of \( C(X) \) is \( z \)-radical.

**PROOF.** Let \( a \) be an \( \Upsilon \)-radical ideal, let \( f \in a \) and let \( g \in C(X) \) with \( A := \{f = 0\} \subseteq \{g = 0\} \). We have to show \( g \in a \) and we may assume that \( f, g \geq 0 \). For \( t \in \mathbb{R} \), \( t > 0 \) let

\[
b(t) := \sup \{g(x) \mid f(x) \leq t\}.
\]
As $X$ is compact, $b(t) \in \mathbb{R}$ and there is some $\xi(t) \in X$ with $f(\xi(t)) \leq t$ and $g(\xi(t)) = b(t)$.

We claim that $\lim_{t \to 0} b(t) = 0$. Since $b : (0, \infty) \to [0, \infty)$ is increasing it is enough to show that $\lim_{n \to \infty} b\left(\frac{1}{n}\right) = 0$. Since $X$ is compact there is some $\xi \in X$ such that for all open sets $U$ containing $\xi$, the set $N(U) := \{ n \in \mathbb{N} \mid \frac{1}{n} < r := f(\xi)/2 \}$ we have $f(\xi) \leq \frac{1}{n} < r$, hence $U = \{ f > r \}$ is a neighborhood of $\xi$ and $N(U)$ is finite.

So $\xi \in A$ and $g(\xi) = 0$, as $A \subseteq \{ g = 0 \}$. Therefore, for all $\varepsilon > 0$, the open set $\{ g < \varepsilon \}$ contains $\xi$ and $N(\{ g < \varepsilon \})$ is infinite, i.e., there are infinitely many $n \in \mathbb{N}$ with

$$b\left(\frac{1}{n}\right) = g(\xi) < \varepsilon.$$ 

Since $b(\frac{1}{n})$ is decreasing as $n \to \infty$ this means that $\lim_{n \to \infty} b\left(\frac{1}{n}\right) = 0$.

By (5.6), there is some increasing homeomorphism $s \in Y$ with $s(t) > b(t)$ on some interval $[0, \delta]$.

We have $g \leq s \circ f$ on $\{ f < \delta \}$. To see this take $x \in X$ with $t := f(x) < \delta$. Then $s(f(x)) = s(t) \geq b(t) \geq g(x)$ by definition of $b(t)$.

Finally, $g \leq s \circ f$ on $\{ f < \delta \}$ implies $g \in \sqrt{(s \circ f)}$ by (5.2)(ii). So $g \in \sqrt{(s \circ f)} \subseteq \sqrt{\alpha} = \alpha$.

(8.2) Definition. We say that a Tychonoff space $X$ has computable $z$-radicals if $\sqrt{\alpha} = \sqrt[\text{rad}]{}$ for all ideals $\alpha$ of $C(X)$.

By (8.1), compact spaces have computable $z$-radicals. Here is a criterion, how to produce new spaces with computable $z$-radicals from given ones.

(8.3) Proposition. Let $\tau : X \to Y$ be a continuous map between Tychonoff spaces $X$, $Y$ and let $\varphi : C(Y) \to C(X)$ be the corresponding ring homomorphism, so $\varphi(g) = g \circ \tau$.

Suppose the following conditions hold:

a. $Y$ has computable $z$-radicals.

b. If $p \in \text{Spec } C(X)$ is $T$-radical, then $\text{Spec } \varphi$ is injective on the set $\{ p^\circ, p, \varphi(p) \}$.

c. The restriction of $\text{Spec } \varphi$ to prime $z$-ideals is a convex map

$$z \rightarrow \text{Spec } C(X) \to z \rightarrow \text{Spec } C(Y).$$

Then also $X$ has computable $z$-radicals.

Proof. By (5.12), it is enough to show that every $T$-radical prime ideal $p$ of $C(X)$ is $z$-radical. Suppose not. Let $\iota := \text{Spec } \varphi$. By b., $\iota(p^\circ) \subseteq \iota(p) \subseteq \iota(\varphi(p))$. By a., $\iota(p)$ is $z$-radical. By c., there must be a prime $z$-ideal $q$ with $p^\circ \subseteq q \subseteq \sqrt{\varphi}$ which is mapped onto $\iota(p)$. But this is not possible, since $p^\circ \subseteq \sqrt{\varphi}$ is a direct specialization of prime $z$-ideals.

Of course, condition b. of (8.3) is satisfied, if $\text{Spec } C(X) \to \text{Spec } C(Y)$ is injective. Recall that $\text{Spec } C(X) \to \text{Spec } C(Y)$ is injective in the following cases:

(b.1) $C(Y) \to C(X)$ is an epimorphism in the category of rings, or
(b.2) $X$ is a cozero set of $Y$.

Condition c. of (8.3), says that the convexity of $\text{Spec } \varphi$ survives if we look at $z$-prime ideals only.

============ LONG VERSION =============

Proof. If $\varphi : A \to B$ is an epimorphism, then $\text{Spec } \varphi : \text{Spec } B \to \text{Spec } A$ is injective: Let $p, q \in \text{Spec } B$ and suppose $\alpha := \varphi^{-1}(p) = \varphi^{-1}(q)$. Then the residue field $\kappa(\alpha)$ embeds into $\kappa(p)$ and into $\kappa(q)$. Choose $\kappa(\alpha)$-embeddings $\varepsilon : \kappa(p) \to \Omega$, $\delta : \kappa(q) \to \Omega$ into a
field $\Omega$. Then the compositions $A \to B \to B/p \to \Omega$ and $A \to B \to B/q \to \Omega$ are the same. Since $A \to B$ is an epimorphism, also the compositions $B \to B/p \to \Omega$ and $B \to B/q \to \Omega$ are the same. But then $p = q$. \hfill $\Box$

If $X \subseteq Y$ is $C^*$-embedded in $Y$, then $\Spec(C(Y)) \to \Spec(C(X))$ is an epimorphism.

**Proof.** Let $\alpha, \beta : \Spec(C(X)) \to R$ be ring homomorphisms with $\alpha \circ \varphi = \beta \circ \varphi$. Let $f \in C(X)$. We have to show that $\alpha(f) = \beta(f)$ We write $f = f_1 \cdot f_2$ with $f_1 = 1 + f_2$ and $f_2 = \frac{1}{1+ f_2}$.

Since $f_2$ is bounded, $f_2$ is in the image of $\varphi$, say $f_2 = \varphi(g)$. By assumption $\alpha(f_2) = \beta(f_2)$. $f_1$ is a unit of $C(X)$ and $f_1^{-1}$ is in the image of $\varphi$. Hence $\alpha(f_1^{-1}) = \beta(f_1^{-1})$. It follows that also $\alpha(f) = \beta(f)$ as desired. \hfill $\Box$

Direct proof that $\Spec(C(X)) \to \Spec(C(Y))$ is injective if $X \subseteq Y$ is $C^*$-embedded in $Y$: If $p, q \in \Spec(C(X))$ and $f \in p \setminus q$, then $\frac{f}{1+f}$ is in $p \setminus q$ is bounded. So any extension $g$ of $\frac{f}{1+f}$ to $Y$ satisfies $g \in \varphi^{-1}(p) \setminus \varphi^{-1}(q)$.

**Note:** According to Robert Raphael, there is a situation $X \to Y$, such that $\Spec(C(Y)) \to \Spec(C(X))$ is an epimorphism, but $X$ is not $C^*$-embedded into $Y$. Namely the “non locally closed standard situation”: $X = \{0\} \cup \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\}$ and $Y = X \cup \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = 0\}$.

If $X$ is a boolean combination of zero sets of $Y$, then $\Spec(C(X)) \to \Spec(C(Y))$ is injective: By assumption $X$ is of the form $X = \bigcup_{i=1}^{n} \{h_i = 0\} \cap \{g_i \neq 0\}$ for some $h_1, \ldots, h_n, g_1, \ldots, g_n \in C(Y)$. Of course we may assume that all the $h_i, g_i$ are bounded. Let $p, q \in \Spec(C(X))$ and let $f \in p \setminus q$. Let $g := g_1 \cdots g_n$, let $X_i := \{h_i = 0\} \cap \{g_i \neq 0\}$ and let $\hat{f} := \frac{f}{1+f} g | X = C(X)$. Let $y \in \overline{X} \setminus X$. We claim that $\lim_{x \to y} \hat{f}(x) = 0$. Since $\overline{X} = \bigcup_{i=1}^{n} \overline{X_i}$ we know that $y \in \overline{X_i}$ for some $i$ and it is enough to show $\lim_{x \to y, x \in X_i} \hat{f}(x) = 0$ for each of these $i$. Since $X_i = \{h_i = 0\} \cap \{g_i \neq 0\}$ and $y \in \overline{X_i} \setminus X_i$ we must have $g_i(y) = 0$. Since $\frac{f}{1+f} (g_1 \cdots g_{i-1} g_{i+1} \cdots g_n) | X$ is bounded on $X$ and $g_i$ is continuous on $Y$, it follows $\lim_{x \to y, x \in X} \hat{f}(x) = 0$.

Thus we have shown $\lim_{x \to y} \hat{f}(x) = 0$ for each $y \in \overline{X}$. This implies that $\hat{f}$ can be extended to a continuous function on $\overline{X}$ and by the Tietze extension theorem (it is applicable to $Y$, since $Y$ is compact) there is a continuous extension $\hat{f}$ of $\hat{f}$ to $Y$. **Problem:** $\hat{f}$ does not extend $f$!!

================================ END OF LONG VERSION ======================

(8.3) allows a reformulation of the property “$X$ has computable z-radicals” in terms of $\beta X$.

First recall how $\Spec(C(X))$ is located in $\Spec(C(\beta X))$:

(8.4) **Remark.** Let $\varphi : C(\beta X) \to C(X)$ be the restriction map. Then

(i) $\Spec(\varphi)$ is an homeomorphism from $\Spec(C(X))$ onto the image of $\Spec(\varphi)$ in $\Spec(C(\beta X))$.

(ii) The image of $\Spec(\varphi)$ is a proconstructible subset of $\Spec(C(\beta X))$, which is closed under generalizations.

(iii) $(\Spec(\varphi))(\mathcal{T} \setminus \Spec(C(X)))$ is a proconstructible subset of $\mathcal{T} \setminus \Spec(C(\beta X))$, which is closed under generalizations.

================================ BEGIN OF LONG VERSION ======================

(iv) If $p$ is a prime ideal of $C(X)$, then $\varphi^{-1}(p)$ is a maximal ideal of $C(\beta X)$ if and only if $C(X)/p \cong \mathbb{R}$.

(v) The Hewitt real compactification $\nu X := \{m \in \beta X \mid C(X)/m \cong \mathbb{R}\} \subseteq \beta X$ of $X$ is the intersection of cozero sets of $\beta X$.

(vi) $(\Spec(\varphi))(\Spec(C(X)))_{\min}$ is an homeomorphism $(\Spec(C(X)))_{\min} \to (\Spec(C(\beta X)))_{\min}$. 

}
(vii) (Gelfand-Kolmogoroff). Let \( r : \text{Spec} C(\beta X) \longrightarrow (\text{Spec} C(\beta X))^{\max} \) be the map \( r(p) = \text{the maximal ideal of } C(\beta X) \text{ which contains } p \). Then the map

\[
r \circ (\text{Spec} \varphi)|_{(\text{Spec} C(X))^{\max}} : (\text{Spec} C(X))^{\max} \longrightarrow (\text{Spec} C(\beta X))^{\max}
\]

is an homeomorphism.

\[=\text{END OF LONG VERSION}===\]

**Proof.** (i) and (ii) is explained in [Schw2], section 5. (iii) holds by (i), (ii) and since \( \text{Spec } \cf(X) \) contained in the image of \( \text{Spec } C(X) \). Conversely suppose \( \varphi : C(\beta X) \longrightarrow C(X) \) is injective. Then \( \varphi : (\text{Spec} C(\beta X))^{\max} \longrightarrow (\text{Spec} C(X))^{\max} \) is an isomorphism for every prime ideal of \( C(X) \).

(i) Since \( X \) is \( C^* \)-embedded into \( \beta X \), \( \varphi \) is an epimorphism and \( \text{Spec } \varphi \) is injective. If \( p \in \text{Spec } C(X) \), then \( p \) is the ideal generated by \( \varphi(\varphi^{-1}(p)) \) in \( C(X) \) (for \( f \in p \) we have \( \frac{1}{1+g} \in p \cap \text{im } \varphi \), hence \( f = (1 + f^2) \cdot \frac{1}{1+g} \) is in the ideal generated by \( \varphi(\varphi^{-1}(p)) \)).

Therefore \( \varphi^{-1}(p) \subseteq \varphi^{-1}(q) \) implies \( p \subseteq q \) for all \( p, q \in \text{Spec } C(X) \). This shows that \( \text{Spec } \varphi \) is an homeomorphism onto the image.

(ii) Since \( \varphi \) is injective, \( \text{Spec } \varphi \) is dominant, hence the minimal points of \( \text{Spec } C(\beta X) \) are contained in the image of \( \text{Spec } \varphi \). By (6.4), (i) implies that the image of \( \text{Spec } \varphi \) is a convex subset of \( \text{Spec } C(\beta X) \). These two assertions are equivalent to (ii).

(iii) holds by (i), (ii) and since \( \text{Spec } \varphi \) is a convex map (cf. (6.5)).

(iv) follows easily from the observation that \( qf(C(\beta X)/\varphi^{-1}(p)) \longrightarrow qf(C(X)/p) \) is an isomorphism for every prime ideal of \( C(X) \).

(v) By (ii), the image of \( \text{Spec } \varphi \) is closed in the inverse topology of \( \text{Spec } C(\beta X) \). Hence there is a subset \( S \subseteq C(\beta X) \) such that \( (\text{Spec } \varphi)(\text{Spec } C(X)) = \bigcap_{f \in S} D(f) \). From (iv) we get

\[
\nu \subseteq \bigcap_{f \in S} D(f) = \bigcap_{f \in S} D(f) \subseteq \bigcap_{f \in S} D(f).
\]

(vi) \( (\text{Spec } C(\beta X))^{\min} \) is in the image of \( (\text{Spec } \varphi)|_{(\text{Spec } C(X))^{\min}} \), since \( \text{Spec } \varphi \) is dominant (the restriction map \( C(\beta X) \longrightarrow C(X) \) is injective !). By (i) we get (vi).

(vii) Since \( r \) is continuous and \( (\text{Spec } C(X))^{\max} \), \( (\text{Spec } C(\beta X))^{\max} \) are compact Hausdorff it is enough to show that \( \kappa := r \circ (\text{Spec } \varphi)|_{(\text{Spec } C(X))^{\max}} \) is a bijection \( (\text{Spec } C(X))^{\max} \longrightarrow (\text{Spec } C(\beta X))^{\max} \).

\( \kappa \) is surjective: let \( m \in (\text{Spec } C(\beta X))^{\max} \). Pick \( p \in (\text{Spec } C(\beta X))^{\min} \) with \( p \subseteq m \). Then \( p \) is in the image of \( \text{Spec } \varphi \) by (vi) and for every \( n \in (\text{Spec } C(X))^{\max} \) containing the preimage of \( p \) under \( \text{Spec } \varphi \) we have \( p \subseteq \varphi^{-1}(n) \subseteq \kappa(n), m \). So \( \kappa(n) = m \).

\( \kappa \) is injective: Let \( m, n \) be different maximal ideals of \( C(X) \). We have to show that \( \kappa(n) \neq \kappa(m) \). Let \( f \in m \) and \( g \in n \) such that \( f + g = 1 \). Then \( f^2 + g^2 \) does not have zeroes and \( \frac{f^2}{1+f^2}, \frac{g^2}{1+g^2} \) are restrictions of functions \( f_1, g_1 \subseteq C(\beta X) \) respectively. Since \( f_1 + g_1 = 1 \) on \( X \) we get \( f_1 + g_1 = 1 \). As \( f_1 \in \kappa(m), g_1 \in \kappa(n) \) we see that \( 1 \in \kappa(m) + \kappa(n) \) as desired.

\[=\text{END OF LONG VERSION}===\]

(8.5) **Corollary.** \( X \) has computable \( z \)-radicals if and only if \( (\text{Spec } \varphi)(z-\text{Spec } C(X)) \) is a convex subset of \( z-\text{Spec } C(\beta X) \). Here \( \varphi : C(\beta X) \longrightarrow C(X) \) denotes the restriction map again.

**Proof.** If \( (\text{Spec } \varphi)(z-\text{Spec } C(X)) \) is a convex subset of \( z-\text{Spec } C(\beta X) \), then \( X \) has computable \( z \)-radicals by (8.1), (8.4) and (8.3).

Conversely suppose \( X \) has computable \( z \)-radicals. Then \( \text{Spec } C(X) = z-\text{Spec } C(X) \) and by (8.4) (iii), \( (\text{Spec } \varphi)(z-\text{Spec } C(X)) \) is a convex subset of \( z-\text{Spec } C(\beta X) \). \( \square \)
(8.6) **Corollary.** $X$ has computable $z$-radicals if and only if the real compactification $\nu X$ of $X$ has computable $z$-radicals.

**Proof.** The restriction map $C(\nu X) \rightarrow C(X)$ is an isomorphism which induces an isomorphism $z$–Spec $C(X) \rightarrow z$–Spec $C(\nu X)$. By (8.5), $X$ has computable $z$-radicals if and only if $\nu X$ has computable $z$-radicals. □

(8.7) **Lemma.** Let $Y$ be a Tychonoff space and let $X$ be a boolean combination of zero sets of $Y$, say $X = \bigcup_{i=1}^{n} \{ f_i = 0 \} \cap \{ g_i \neq 0 \}$ for some $f_1, \ldots, f_n, g_1, \ldots, g_n \in C(Y)$. Then the induced map $\pi : z$–Spec $C(X) \rightarrow z$–Spec $C(Y)$ is an homeomorphism onto $\bigcup_{i=1}^{n} V(f_i) \cap D(g_i) \cap z$–Spec $C(Y)$. Moreover $\pi$ is convex if and only if $X$ is of the form $\{ f = 0 \} \cap \{ g \neq 0 \}$ for some $f, g \in C(Y)$.

**Proof.** Let $L_X, L_Y$ be the lattices of zero sets of $X$ and $Y$, respectively. Then $\pi$ is induced by the lattice homomorphism $L_Y \rightarrow L_X$, which sends $S$ to $S \cap X$. Since $X$ is the boolean combination of elements from $L_Y$, the Stone-duality for spectral spaces and distributive lattices implies that $\pi$ is an homeomorphism $z$–Spec $C(X) \rightarrow \bigcup_{i=1}^{n} V(f_i) \cap D(g_i) \cap z$–Spec $C(Y)$.

Since $\pi$ is an homeomorphism onto the image, $\pi$ is convex, if and only if the image of $\pi$ is a convex subset of $z$–Spec $C(Y)$. Since the image of $\pi$ is a constructible subset of $z$–Spec $C(Y)$, this image is convex if and only if it is of the form $A \setminus B$ for some closed constructible subsets $A, B$ of $z$–Spec $C(Y)$ (by (6.2)). Since the closed constructible subsets of $z$–Spec $C(Y)$ are precisely the sets of the form $V(f) \cap z$–Spec $C(Y)$, we get the lemma. □

Remark. In (i) of (8.7) we do not claim that $X$ is $z$-embedded into $Y$.

(8.8) **Corollary.** Let $Y$ be a Tychonoff space

(i) If $X_1, X_2$ are boolean combinations of zero sets of $Y$ with $X_1 \cap X_2 = \emptyset$, then there is a constructible subset $K$ of Spec $C(Y)$ with $X_1 \subseteq K$ and $K \cap X_2 = \emptyset$.

(ii) If $K$ is a constructible subset of Spec $C(Y)$, $C(Y) \rightarrow C(X)$, then the image of Spec $\varphi$ is the convex hull of $K \cap z$–Spec $C(Y)$.

**Proof.**

**UNDER CONSTRUCTION:** □

Recall that $X \subseteq Y$ is $z$-embedded into $Y$ if every zero set of $X$ is of the form $Z \cap X$ for some zero set of $Y$.

(8.9) **Theorem.** Let $Y$ be a Tychonoff space with computable $z$-radicals. Suppose $X \subseteq Y$ is of the form $\{ f = 0 \} \cap \{ g \neq 0 \}$ for some $f, g \in C(Y)$ and $\{ f = 0 \}$ is $C^*$-embedded in $Y$.

Then $X$ has computable $z$-radicals.

**Proof.** By (8.7), condition c. of (8.3) is satisfied. Since $X$ is a cozero set of $\{ f = 0 \}$, Spec $C(X) \rightarrow$ Spec $C(\{ f = 0 \})$ is injective. By assumption, $\{ f = 0 \}$ is $C^*$-embedded in $Y$, hence Spec $C(\{ f = 0 \}) \rightarrow$ Spec $C(Y)$ is injective. Therefore, condition b. of (8.3) is satisfied for $X \rightarrow Y$, too.

Hence by (8.3), $X$ has computable $z$-radicals. □
(8.10) Corollary. Let $Y$ be pseudo compact and let $X$ be a cozero set of $Y$. Then $X$ has computable $z$-radicals.

Proof. If $Y$ is pseudo compact, then $\nu Y$ is compact. By (8.1) and (8.6), $Y$ has computable $z$-radicals. Hence by (8.9), $X$ has computable $z$-radicals. □

Up to now we have not seen spaces without computable $z$-radicals. In order to see a lot of them we first introduce a method which produces $\Upsilon$-radical ideals of $C(X)$.

(8.11) Proposition. (Production of $\Upsilon$-radical ideals)

Let $Y$ be a Tychonoff space and let $X \subseteq Y$. Let $\mathfrak{f}$ be a proper filter of subsets of $Y$ such that for every $g \in C(X)$ there is a set $O \in \mathfrak{f}$, open in $Y$ so that $g$ is bounded on $O \cap X$. Then the set

$$a(f) := \{ f \in C(X) \mid \{ y \in \overline{X} \mid \lim_{x \rightarrow y} f(x) = 0 \} \in \mathfrak{f} \}$$

is a proper $\Upsilon$-radical ideal of $C(X)$.

Proof. Since $\mathfrak{f}$ is closed under finite intersections we have $a(f) + a(f) \subseteq a(f)$. Now let $g \in C(X)$ and let $f \in a(f)$. Since $f \in a(f)$, the set $Z := \{ y \in \overline{X} \mid \lim_{x \rightarrow y} f(x) = 0 \}$ is in $\mathfrak{f}$. By assumption there is some $O \in \mathfrak{f}$, open in $Y$ such that $g$ is bounded on $O \cap X$, say $|g| < c$ on $O \cap X$. Then for $y \in Z \cap O$ we have $\lim_{x \rightarrow y} g(x)f(x) \leq \lim_{x \rightarrow y} |g(x)|f(x) = 0$, thus $\{ y \in \overline{X} \mid \lim_{x \rightarrow y} g(x)f(x) = 0 \} \subseteq Z \cap O \in \mathfrak{f}$ and $g \cdot f \in a(f)$.

This shows that $a(f)$ is an ideal of $C(X)$. Since $\mathfrak{f}$ is proper, it does not contain the empty set, thus $1 \notin a(f)$ and $a(f)$ is proper.

If $f \in a(f)$ and $s \in \Upsilon$, then clearly $s(f) \in a(f)$, hence $a(f)$ is an $\Upsilon$-radical ideal of $C(X)$. □

Here are two examples where (8.11) is applicable:

(8.12) Corollary. Let $Y$ be a Tychonoff space and let $\emptyset \neq K \subseteq X \subseteq Y$, such that $K$ is bounded in $X$ (cf. (9.1)). Let $\mathfrak{f}$ be a proper filter of subsets of $Y$ such that one of the following conditions holds:

1. $\mathfrak{f}$ contains the neighborhoods of $K$ in $Y$, or
2. $X$ is $z$-embedded into $Y$ and $\mathfrak{f}$ contains the neighborhoods of $K$ in $Y$ that are cozero sets in $Y$.

Then the set

$$a(f) := \{ f \in C(X) \mid \{ y \in \overline{X} \mid \lim_{x \rightarrow y} f(x) = 0 \} \in \mathfrak{f} \}$$

is a proper $\Upsilon$-radical ideal of $C(X)$.

Proof. We want to apply (8.11) and we have to check the assumption on $\mathfrak{f}$ there. So let $g \in C(X)$. Since $K$ is bounded in $X$, there is some $c \in \mathbb{R}$ with $|g| < c$ on $K$, and the set $U := \{|g| < c\}$ is a cozero set of $X$ containing $K$. Now each of the assumptions 1. and 2. implies that $\mathfrak{f}$ contains an open subset $O$ of $Y$ with $O \cap X = U$. This shows that (8.11) is applicable. □

The next proposition says that a space $X$ with computable $z$-radicals forces a certain location of $X$ inside every space $Y$ containing it. We’ll use this configuration later to produce spaces without computable $z$-radicals.

(8.13) Proposition. ($\Upsilon$-configuration lemma.)

Let $Y$ be a Tychonoff space and let $Z, X \subseteq Y$ such that the following conditions hold:

a. $Z$ and $Z \cap X$ are zero sets of $Y$.

b. $Z \cap X$ is bounded in $X$ and $X$ has computable $z$-radicals.
Then $Z \cap X$ is open in $Z \cap \overline{X}$. If $X$ is $z$-embedded in $Y$, then there is some $f \in C(Y)$ such that $Z \cap X = \{f \neq 0\} \cap Z \cap \overline{X}$.

**Proof.** Let $K := Z \cap X$. Let $\mathcal{f}$ be the following filter of subsets of $Y$: If $X$ is $z$-embedded in $Y$ we take $\mathcal{f}$ as the filter of subsets of $Y$ generated by the sets $U \cap Z \cap \overline{X}$, where $U$ is a cozero set of $Y$ containing $K$.

If $X$ is not $z$-embedded in $Y$ we take $\mathcal{f}$ as the filter of subsets of $Y$ generated by the sets $U \cap Z \cap \overline{X}$, where $U$ is an open neighborhood of $K$ in $Y$.

Then $\mathcal{f}$ satisfies the assumptions of (8.12) (note that by b., $K = Z \cap X$ is bounded in $X$) and the ideal $a(\mathcal{f})$ defined there is $\mathcal{T}$-radical. By a., there are $h, g \in C(Y)$ such that $\{h = 0\} = Z$ and $\{g = 0\} = Z \cap X$. Since $Z \cap \overline{X} \in \mathcal{f}$ we have $h|_X \in a(\mathcal{f})$ (cf. the definition of $a(\mathcal{f})$). Moreover we have $\{g|_X = 0\} = Z \cap X = \{h|_X = 0\}$.

By b., $a(\mathcal{f})$ is $\mathcal{T}$-radical, thus $g|_X \in a(\mathcal{f})$. By definition of $\mathcal{f}$ and $a(\mathcal{f})$ we must have $U \cap Z \cap \overline{X} \subseteq Z \cap X$ for some open neighborhood $U$ of $K$ in $Y$, which can be chosen to be a cozero set of $Y$ if $X$ is $z$-embedded in $Y$. Hence $U \cap Z \cap \overline{X} \subseteq Z \cap X = K \subseteq U \cap Z \cap \overline{X}$ and $Z \cap X = U \cap Z \cap \overline{X}$ as claimed.

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**LONG VERSION**

(8.14) **Proposition.** Let $Y$ be a Tychonoff space and let $X \subseteq Y$ be such that $X$ has computable $z$-radicals. Then $X$ is locally closed in $Y$.

**Proof.** Suppose $X$ is not locally closed in $Y$. This means that there is some $x \in X$ such that for each open neighborhood $U$ of $x$ in $Y$, the set $X \cap U$ is not closed in $U$. In other words $x$ is in the closure of $\overline{X} \setminus X$. Let

$$a := \{f \in C(X) \mid x \in \{y \in \overline{X} \setminus X \mid \lim_{t \to y} f(t) = 0\}\}.$$ 

Then $a$ is a $\mathcal{T}$-radical ideal of $C(X)$ (proof as in (8.17)).

Why is $a$ not $z$-radical?

**UNDER CONSTRUCTION:**

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(8.15) **Corollary.** Let $Y$ be compact and let $X$ be a boolean combination of zero sets of $Y$. If $X$ has computable $z$-radicals, then $X$ is locally closed in $Y$.

**Proof.** Since $X$ is a boolean combination of zero sets of $Y$, there are $f_1, \ldots, f_n, g_1, \ldots, g_n \in C(Y)$ such that

$$X = \bigcap_{i=1}^n (\{f_i = 0\} \cup \{g_i \neq 0\}).$$

By replacing $f_i$ with $f_i^2 + g_i^2$ if necessary, we may assume that $\{f_i = 0\} \cap \{g_i \neq 0\} = \emptyset$. Let $x \in X$. We have to find an open neighborhood $O$ of $x$ in $Y$ such that $X \cap O$ is closed in $O$. Let $I := \{i \in \{1, \ldots, n\} \mid f_i(x) = 0\}$ and let $\tilde{f} := \prod_{i \in \{1, \ldots, n\} \setminus I} f_i$. Then $\tilde{f}(x) \neq 0$ and $x \in X \cap \{\tilde{f} \neq 0\}$, which is a cozero set of $X$. By (8.14), $X \cap \{\tilde{f} \neq 0\}$ has computable $z$-radicals again and we may replace $X$ by $X \cap \{\tilde{f} \neq 0\}$. Since $\{\tilde{f} \neq 0\} = \bigcap_{i \in \{1, \ldots, n\} \setminus I} \{f_i \neq 0\}$, we obtain a set $S$ so that

$$x \in X = \bigcap_{i=1}^n (\{f_i = 0\} \cup \{g_i \neq 0\}) \cap \{g \neq 0\}$$

and $f_i(x) = g_i(x) = 0$ for all $i \in \{1, \ldots, n\}$.

Since $g(x) \neq 0$, there is a zero set $N$ of $Y$ such that $N \subseteq \{g \neq 0\}$ and such that $x$ is in the interior of $N$ (take $c := g(x)$, $\varepsilon := \frac{1}{2}|c| \in (0, +\infty)$ and take $N := \{c - \varepsilon \leq g \leq c + \varepsilon\})$. 

Take $A := \bigcap_{i=1}^{n} \{ f_i = 0 \}$, $B := \bigcap_{i=1}^{n} \{ g_i = 0 \}$ and $Z := B \cap N$. Since $\{ f_i = 0 \} \subseteq \{ g_i = 0 \}$ we have $A \subseteq B$. Then $X \cap Z = \bigcap_{i=1}^{n} (\{ f_i = 0 \} \cup \{ g_i \neq 0 \}) \cap B \cap N = A \cap N$. So $Z$ and $Z \cap X$ are zero sets of $Y$. Since $Y$ is compact, also $Z \cap X$ is compact and since $X$ has computable $z$-radicals, (8.13) says that $Z \cap X$ is open in $Z \cap Y$. Let $O \subseteq Y$ be open such that $A \cap N = O \cap B \cap N \cap Y$.

**UNDE R CONSTRUCTION:**

(8.16) Corollary. Let $Y$ be compact and let $f, g \in C(Y)$. Let $X := \{ f = 0 \} \cup \{ g \neq 0 \}$. If $X$ has computable $z$-radicals, then $X$ is locally closed. If in addition $X$ is $z$-embedded into $Y$, then there is a cozero set $O$ of $Y$ such that $X = \overline{X} \cap O$.

**Proof.** Let $Z := \{ f: g = 0 \} = \{ f = 0 \} \cup \{ g = 0 \}$. Then $Z \cap X = \{ f = 0 \}$, so $Z$ and $Z \cap X$ are zero sets of $Y$. Since $Y$ is compact, $Z \cap X$ is compact. As $X$ has computable $z$-radicals, (8.13) gives an open subset $U$ of $Y$ such that $\{ f = 0 \} = Z \cap X = U \cap Z \cap \overline{X}$, moreover if $X$ is $z$-embedded in $Y$ we may assume that $U$ is a cozero set of $Y$.

Since $Z \cap \overline{X} = (\{ f = 0 \} \cup \{ g = 0 \}) \cap (\{ f = 0 \} \cup \{ g \neq 0 \}) = \{ f = 0 \} \cup \partial \{ g \neq 0 \}$, we get $\{ f = 0 \} = U \cap \{ f = 0 \} \cup \partial \{ g \neq 0 \}$ and this easily implies $X = \{ f = 0 \} \cup \{ g \neq 0 \} = \overline{X} \cap (U \cup \{ g \neq 0 \})$.

So with $O := U \cup \{ g \neq 0 \}$ we have $X = \overline{X} \cap O$ is locally closed, and $O$ is a cozero set of $Y$ if $X$ is $z$-embedded into $Y$. □

For example the set $X := \{(x, y) \in [0, 1]^2 \mid x, y > 0 \} \cup \{(0, 0)\} \subseteq [0, 1]^2 \subseteq \mathbb{R}^2$ does not have computable $z$-radicals as it is not locally closed in $[0, 1]^2$. Actually, in metric spaces, only locally closed subsets can have computable $z$-radicals:

(8.17) Corollary. Let $Y$ be a metric space and let $X \subseteq Y$ be such that $X$ has computable $z$-radicals. Then $X$ is locally closed in $Y$.

**Proof.** Suppose $X$ is not locally closed in $Y$. This means that there is some $x \in X$ such that for each $\varepsilon > 0$ the set $X \cap B_{\varepsilon}(x)$ is not closed in the open ball $B_{\epsilon}(x)$ of radius $\varepsilon$ around $x$. Thus we can construct a sequence $(y_k)_{k \in \mathbb{N}} \in \overline{X} \setminus X$ which converges to $x$. Let $Z := \{ y_k \mid k \in \mathbb{N} \} \cup \{ x \}$. Then $Z \subseteq \overline{X}$ is closed and therefore a zero set of $Y$ contained in $\overline{X}$. Moreover $Z \cap X = \{ x \}$ is a zero set of $Y$, which is compact and not open in $Z = Z \cap \overline{X}$. So by (8.13), $X$ can not have computable $z$-radicals. □

(8.18) Theorem. A subset $X$ of $\mathbb{R}^n$ has computable $z$-radicals if and only if $X$ is locally closed.

**Proof.** If $X$ has computable $z$-radicals, then by (8.17), $X$ is locally closed.

Conversely let $X \subseteq \mathbb{R}^n$ be locally closed. Since $\mathbb{R}^n$ is homeomorphic to the open unit ball of $\mathbb{R}^n$ we may assume that $X$ is bounded. Since $X$ is locally closed it is of the form $A \cap O$ for some $A \subseteq \mathbb{R}^n$ closed, $O \subseteq \mathbb{R}^n$ open. Then $Y := \overline{X}$ is compact and $X$ is the cozero set of the distance function of $\overline{X} \setminus X$. So by (8.10), $X$ has computable $z$-radicals. □

9. Computation of the diamond for principal ideals

If a Tychonoff space $X$ has computable $z$-radicals and $a$ is an ideal of $C(X)$, then clearly $a^2 = a^2$. In this sense also the ideals $a^n$ are computable.
If \( X \) does not have computable \( z \)-radicals, then there must be an ideal \( \mathfrak{a} \) of \( C(X) \) with \( \mathfrak{a}^ \circ \not\subseteq \mathfrak{a}^T \). In this section we show that under a mild condition on the topological space \( X \), the ideals \( \mathfrak{a}^ \circ \) and \( \mathfrak{a}^T \) still coincide if \( \mathfrak{a} \) is finitely generated (cf. (9.2)). For example \( \mathfrak{a}^ \circ = \mathfrak{a}^T \) holds for all principal ideals \( C(X) \) whenever \( X \) is locally compact or a metric space (recall from (8.17) that many metric spaces do not have computable \( z \)-radicals).

In (9.6) below, a normal space \( X \) and a principal ideal \( \mathfrak{a} \) of \( C(X) \) with \( \mathfrak{a}^ \circ \neq \mathfrak{a}^T \) is constructed.

(9.1) Definition. Let \( X \) be a Hausdorff space. A subset \( B \) of \( X \) is called bounded in \( X \) if every \( h \in C(X) \) is bounded on \( B \).

For example a sequence \( (x_n) \) is bounded in \( X \) if it converges to a point in \( X \) or if all but finitely many of the \( x_n \) are contained in a pseudo compact subset \( Y \) of \( X \); recall that a pseudo compact space is a space \( X \), such that every continuous function on \( X \) is bounded. Examples of pseudo compact spaces that are not compact can be found in [Gil-Jer], 5I.

(9.2) Theorem. Let \( X \) be a Tychonoff space such that the following condition holds:

\[
\text{For every } f \in C(X) \text{ and every cozero set } U \text{ contained in } \{ f \neq 0 \} \text{ with } U \cap \{ f = 0 \} \neq \emptyset \text{ there is a sequence } (x_n)_{n \in \mathbb{N}} \text{ in } U, \text{ bounded in } X, \text{ such that } \lim_{n \to \infty} f(x_n) = 0.
\]

Let \( g \in C(X) \) and let \( \mathfrak{a} \) be an ideal of \( C(X) \) with \( \sqrt{\mathfrak{a}} = \sqrt{\langle g \rangle} \). Then

\[
\mathfrak{a}^T = \mathfrak{a}^\circ = \{ f \in C(X) \mid \{ g = 0 \} \subseteq \text{int} \{ f = 0 \} \}.
\]

Proof. By replacing \( g \) with \( g^2 \) if necessary, we may assume that \( g \geq 0 \). Let \( \mathfrak{b} := \{ f \in C(X) \mid \{ g = 0 \} \subseteq \text{int} \{ f = 0 \} \} \). By (5.3), we know that \( \mathfrak{b} \subseteq (\mathfrak{a})^\circ \subseteq \mathfrak{a}^T \) and we must prove \( \mathfrak{a}^T \subseteq \mathfrak{b} \).

Let \( f \in \mathfrak{a}^T \). We have to show \( \{ g = 0 \} \subseteq \text{int} \{ f = 0 \} \) and we may assume that \( f \geq 0 \). Since \( \{ g = 0 \} \subseteq \{ f = 0 \} \) and \( g \land f \in \mathfrak{a}^\circ \) we may assume that \( 0 \leq f \leq g \), too.

Suppose there is a zero of \( g \), which is not in the interior of \( \{ f = 0 \} \). Then \( U = \{ f \neq 0 \} \) is a cozero set contained in \( \{ g \neq 0 \} \) with \( \overline{U} \cap \{ g = 0 \} \neq \emptyset \). Since \( X \) satisfies condition (\( \star \)), there is a sequence \( (x_n)_{n \in \mathbb{N}} \subseteq U \) with \( \lim_{n \to \infty} g(x_n) = 0 \) such that each \( h \in C(X) \) is bounded on \( \{ x_n \mid n \in \mathbb{N} \} \).

Then \( f(x_n) \neq 0 \) for all \( n \) and \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = 0 \). Of course we can take a subsequence of \( (x_n)_{n \in \mathbb{N}} \) if necessary, with \( f(x_1) > f(x_2) > ... \).

Let \( L : \mathbb{R} \to \mathbb{R} \) be a barrier function (cf. (7.1)). Then there is a continuous function

\[
s : \mathbb{R} \to \mathbb{R} \text{ with } \{ s = 0 \} = \{ 0 \} \text{ and } s(f(x_n)) \geq L(g(x_n)) \text{ for all } n \in \mathbb{N}.
\]

Take \( s \) on \( (0, \infty) \) so that \( s \) is linear on \( [f(x_{n+1}), f(x_n)] \) with \( s(f(x_n)) = L(g(x_n)) \) - this is possible as \( f(x_1) > f(x_2) > ... \). Since \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = 0 \), we get \( \lim_{t \to 0} s(t) = 0 \) and \( s \) can be extended to \( \mathbb{R} \) as desired.

Because \( f \in \mathfrak{a}^T \) we have \( s \circ f \in \mathfrak{a} \subseteq \sqrt{\langle g \rangle} \), and there are some \( h \in C(X) \) and some \( k \in \mathbb{N} \)
with \((s \circ f)^k = g \cdot h\). We may assume that \(k\) is odd and get \(s \circ f = g^{\frac{k}{2}} \cdot h^{\frac{k}{2}}\). Consequently
\[
0 < L(g(x_n)) \leq s(f(x_n)) = g^{\frac{k}{2}}(x_n) \cdot h^{\frac{k}{2}}(x_n) \text{ for all } n \in \mathbb{N}.
\]
Since \(h\) is bounded on \(\{x_n \mid n \in \mathbb{N}\}\), there is some \(N \in \mathbb{N}\) with
\[
0 < L(g(x_n)) \leq s(f(x_n)) \leq g^{\frac{k}{2}}(x_n) \cdot N \text{ for all } n \in \mathbb{N}.
\]
But \(\lim_{n \to \infty} g(x_n) = 0\), which contradicts the barrier condition of (7.1) for \(L\). This shows that \(a^T \subseteq b\). \(\square\)

Before we draw consequences from (9.2), let us look at examples where condition (\(\star\)) is satisfied.

\[\text{LONG VERSION}\]

(9.3) \text{Remark. Let } I \text{ be a totally ordered set of indices. For } i \in I \text{ let } x_i \in \mathbb{R}, x_i > 0 (i \in I) \text{ such that } \lim_{i \in I} x_i = 0. \text{ Then there is a countable sequence } i_1 < i_2 < \ldots \text{ in } I \text{ which is cofinal in } I. \text{ In particular } \lim_{n \to \infty} x_{i_n} = 0.
\]

\text{Proof.} For \(n \in \mathbb{N}\), there is some \(i_n \in I\) such that \(x_i < 1/n\) for all \(i \in I, i_n \leq i\). If the set \(\{i_n \mid n \in \mathbb{N}\}\) is not cofinal in \(I\), then there is some \(i \in I\) with \(i_n \leq i\) for all \(n \in \mathbb{N}\). So \(x_i = 0\) in contradiction to our assumption. \(\square\)

Note that the remark above is not true anymore if we replace “totally ordered index set” by “filtered family”!

\[\text{END OF LONG VERSION}\]

(9.4) \text{Examples. Let } X \text{ be a Tychonoff space.}

(i) If every point in \(X\) has a neighborhood that is bounded in \(X\), then \(X\) clearly satisfies (\(\star\)). In particular locally compact spaces satisfy (\(\star\)).

(ii) Suppose for every cozero set \(U\) of \(X\) and each \(x \in U \setminus U\) there is a totally ordered index set \(I\) and points \(x_i \in U\) such that \(\lim_{i \in I} x_i = x\). Then \(X\) satisfies (\(\star\)).

To see this, take \(f\) and \(U\) as assumed in (\(\star\)), hence \(f\) has a zero \(x\) on the boundary of \(U \subseteq \{f \neq 0\}\). Take a totally ordered index set \(I\) and points \(x_i \in U\) such that \(\lim_{i \in I} x_i = x\). Then \(\lim_{i \in I} f(x_i) = f(x) = 0\). Since \(I\) is totally ordered and \(f(x_i) \neq 0\) for each \(i\), it is easy to see that \(I\) must have cofinal subset \(J\) of order type \(\mathbb{N}\). Then the sequence \((x_j)_{j \in J}\) converges to \(x\), \(\lim_{j \in J} f(x_j) = f(x) = 0\) and \(\{x_j \mid j \in J\}\) is bounded in \(X\).

(iii) From (ii) we get the following examples of Tychonoff spaces which satisfy (\(\star\)):

(a) If every point in \(X\) has a countable basis of neighborhoods, then \(X\) satisfies (\(\star\)).

(b) Totally ordered sets equipped with the order topology satisfy (\(\star\)).

(c) Subspaces of a finite power of a totally ordered abelian group satisfy (\(\star\)).

\[\text{LONG VERSION}\]

(b) Let \(T_1, \ldots, T_n\) be totally ordered sets, each one equipped with the order topology and let \(X \subseteq T_1 \times \ldots \times T_n\). Then \(X\) satisfies (\(\star\)):

It is a combinatorial exercise to show that for every cozero set \(U\) of \(X\) and each \(x \in U \setminus U\) there is a totally ordered index set \(I\) and points \(x_i \in U\) with \(\lim_{i \in I} x_i = x\). We leave it to the reader. Observe that it is crucial to assume that \(x\) is an accumulation point of a cozero set of \(X\) here; in general, not every point of \(T_1 \times \ldots \times T_n\) is the proper limit of a sequence with totally ordered index set.

\text{Proof. Claim 1. Let } I \text{ be an upward directed Poset. Then every cofinal subset of } I \text{ is again upward directed (here, by cofinal we mean that for every } i \in I \text{ there is an element } j \text{ in the set with } i \leq j\). If } I = J \cup K, \text{ then } J \text{ or } K \text{ is cofinal in } I.
Proof. This is obvious.

Claim 2. Let \((x_i)_{i \in I}\) be a net in a topological space \(X\) converging to a point \(x \in X\).

Let \(I = I_1 \cup \ldots \cup I_n\). Then for some \(k \in \{1, \ldots, n\}\), \(I_k\) is upward directed and \((x_i)_{i \in I_k}\) converges to \(x\).

Proof. By induction we may assume that \(k = 2\) and \(I = J \cup K\). By claim 1, we may assume that \(J\) is cofinal in \(I\). Then \(J\) is upward directed and clearly \((x_i)_{i \in J}\) converges to \(x\).

We prove the assertion by induction on \(n\): if \(n = 1\) this is obvious.

Assume \(X \subseteq T_1 \times \ldots \times T_n\). Let \(U \subseteq X\) be a cozero set and let \(x \in U \setminus U\).

Let \(f \in C(X)\) with \(\{f \neq 0\} = U\). Let \((x_i)_{i \in I} \subseteq U\) be a net converging to \(x\). We may assume that \(I\) is the set of all open boxes containing \(x\) and that \(x_i \in i\). We write \(x_i = (x_i^1, \ldots, x_i^n)\) and \(x = (x^1, \ldots, x^n)\) with \(x_i^j \in T_j\). By claim 2 above we may assume that for each \(j\), either \(x_i^j < x^j\) (\(i \in I\)) or \(x_i^j = x^j\) (\(i \in I\)) or \(x_i^j > x^j\) (\(i \in I\)).

Case 1. There is some \(j \in \{1, \ldots, n\}\) such that \(x_i^j = x^j\) (\(i \in I\)). Say \(j = n\).

Let \(Y := \{t^1, \ldots, t^n\} \in X\). Then \(x_i, x_i \in Y\) for all \(i\) and \(Y\) is homeomorphic to a subset of \(T_1 \times \ldots \times T_{n-1}\triangleq Y\). Then \(x \in Y\) and for some \(i \in I\) we have \(x_i \in \bigcap Y\).

The index \(i\) is an open box \(\prod_{j=1}^n (a^j, b^j)\) and \(x_i \in \prod_{j=1}^n (a^j, b^j)\). Hence whenever \(k \in \mathbb{N}\) with \(x_i^k > O\), then \(x_i^k \notin X\) and so \(i \notin I_k\).

**Case 2.** There is no \(j \in \{1, \ldots, n\}\) such that \(x_i^j = x^j\) (\(i \in I\)).

We may assume that \(x_i^j > x^j\) (\(i \in I\)). For each \(k \in \mathbb{N}\) pick \(i_k \in I\) such that \(\|f(x_i^k)\| \leq \frac{1}{k}\) (\(i \geq i_k\)).

Claim. For each \(j \in \{1, \ldots, n\}\), the sequence \((x_i^k)_{k \in \mathbb{N}}\) converges to \(x^j\).

Proof. Suppose not. Then for some \(j\), there is an open interval \(O\) of \(T_j\) such that for infinitely many \(k \in \mathbb{N}\) the \(x_i^k\) are strictly bigger than \(O\). Say \(j = n\).

Let \(V := T_1 \times \ldots \times T_{n-1} \times O\). Then \(x \in V\) and for some \(i \in I\) we have \(x_i \in V\).

The index \(i\) is an open box \(\prod_{j=1}^n (a^j, b^j)\) and \(x_i \in \prod_{j=1}^n (a^j, b^j)\). Hence whenever \(k \in \mathbb{N}\) with \(x_i^k > O\), then \(x_i k \notin V\) and so \(i \notin i_k\).

**UNDER CONSTRUCTION: wie weiter?**
(a) $X$ is a normal Hausdorff space (in particular $X$ is a Tychonoff space).

$X$ is Hausdorff, since all the sets $\{x\}$, $x \neq (0,0)$ are clopen. If $A, B$ are closed disjoint subsets of $X$, say $(0,0) \not\in A$, then $A$ is also open, hence the characteristic function of $A$ is continuous. This shows that $X$ is normal.

(b) Clearly, the point $(0,0)$ is the unique non-isolated point of $X$. Hence a function $X \to \mathbb{R}$ is continuous if and only if it is continuous in $(0,0)$.

(c) If $I$ is a totally ordered index set and $x_i \in X \setminus \{(0,0)\}$, then $(0,0)$ is not the limit of $(x_i)_{i \in I}$.

Suppose $\lim_{i \in I} x_i = (0,0)$. Let $\kappa$ be the cofinality of $I$. We first claim that $\kappa = \omega$.

Let $\varphi : I \to S := \{x_i \mid i \in I\}$ be defined by $\varphi(i) = x_i$. Since $S$ is countable and $\kappa$ is uncountable, there is some $s \in S$ such that $\varphi^{-1}(s)$ is cofinal in $I$. But then $(x_i)_{i \in I}$ can not converge to $(0,0)$, since the complement of $s$ in $X$ is an open neighborhood of $(0,0)$.

This shows that $\kappa = \omega$. Let $i_0 < i_1 < \ldots \in \kappa$ be a cofinal subsequence. Since $\lim_{i \in I} x_i = (0,0)$, also $\lim_{i \in I} x_{i_n} = (0,0)$. Thus we may assume $I = \omega$ and there is a sequence $(x_i)_{i \in \omega} \subseteq X \setminus \{(0,0)\}$ with $\lim_{i \to \omega} x_i = (0,0)$. Let $S := \{x_i \mid i < \omega\}$. Fix $t \in T$. Then $S \cap \{(t) \times T\}$ is finite, otherwise $(0,0) \cup [(T \setminus \{t\}) \times T]$ is an open neighborhood of $(0,0)$ which does not contains infinitely many $x_i$ in contradiction to $\lim_{i \to \omega} x_i = (0,0)$.

Since $S \cap \{(t) \times T\}$ is finite for each $t$, the set $\{(0,0)\} \cup (X \setminus S)$ is an open neighborhood of $(0,0)$ which again contradicts $\lim_{i \to \omega} x_i = (0,0)$.

(d) If $S \subseteq X$ is infinite, then there is an infinite subset $S'$ of $S$, clopen in $X$ consisting of isolated points.

Of course we may assume that $(0,0) \not\in S$. If there is some $t \in T$ such that $S \cap \{(t) \times T\}$ is infinite, then this set has the required properties. Hence we may assume that $S \cap \{(t) \times T\}$ is finite for each $t \in T$. Then $X \setminus S$ is open. Hence $S'$ itself has the required property.

(e) All bounded subsets of $X$ are finite.

To see this take an infinite set $S \subseteq X$. By the previous item, there is an infinite subset $S'$ of $S$, clopen in $X$, consisting of isolated points. Hence any unbounded function on $S'$ can be extended to a continuous function on $X$. In particular, $S$ is not bounded.

(f) $X$ is not sequential.

To see this, let $(y_k)_{k \in \mathbb{N}}$ be an enumeration of $X \setminus \{(0,0)\}$. Let $\chi_k$ be the characteristic function of $\{y_k\}$. $\chi_k$ is continuous as $y_k \neq (0,0)$ is an isolated point.

Therefore the function $f(x) := \sum_{k \in \mathbb{N}} \chi_k(x)$ defined on $X$ is continuous and $(0,0)$ is the unique zero of $f$. In other words, with $U := \{f \neq 0\}$ we have $U \cap (0,0) = 0$. But the previous item says that no infinite subset of $U$ if bounded. In particular there is no sequence $(x_n)_{n \in \mathbb{N}}$ in $U$, bounded in $X$, such that $\lim_{n \to \infty} f(x_n) = 0$.

(g) Let $T = \mathbb{N}_0$ and let $g : X \to \mathbb{R}$ be defined by

$$g(m, n) = \begin{cases} \max\{|m|, |n|\} & \text{if } (m, n) \neq (0,0) \\ 0 & \text{if } m = n = 0. \end{cases}$$

Let $f : X \to \mathbb{R}$ be defined by $f(m, n) = g(m, n)$. Then $g$ is continuous, since $g < \frac{1}{k}$ on $\{(0,0)\} \cup \{(k+1, k+2, \ldots) \times T\}$ and this set is a neighborhood of $(0,0)$. $f$ is continuous, since $f < \frac{1}{k}$ on $\{(0,0)\} \cup \{(m, n) \mid n > k\}$ and this set is a neighborhood of $(0,0)$, too. We claim that $f \in (g)^T \setminus \sqrt{(g)}^p$.

First suppose $f \in [\sqrt{(g)}]^p$. Take a barrier function $L$. Then $\{L \circ g = 0\} = \{g = 0\} = \{(0,0)\} = \{f = 0\}$. Hence $f \in [\sqrt{(g)}]^p$ implies $L \circ g \in \sqrt{(g)}^p \subseteq \sqrt{(g)}$. So $(L \circ g)^k = g \cdot h$.
for some $h \in C(X)$ and some $k \in \mathbb{N}$. Since $h$ is continuous, there is an open subset $U$ of $X$ containing $(0,0)$ such that $|h(x)| < |h((0,0))| + 1 =: M$. By definition of the topology of $X$, there is a sequence $(m_i, n_i)_{i \in \mathbb{N}} \subseteq U$ such that $1 \leq m_1 < m_2 < \ldots$ and $\lim_{i \to \infty} m_i = \infty$. Then $L(h) = L(g(m_i, n_i)) = g(m_i, n_i) - h(m_i, n_i) \leq \frac{1}{m_i} \cdot M$ for all $i$, which contradicts the barrier condition for $L$. This shows that $f \not\in \sqrt{(g)^\circ}$.

In order to prove $f \in (g)^\circ$ it is enough to show that $g$ divides $s \circ f$ for every strictly increasing homeomorphism $s \in \mathcal{Y}$. We define $h : X \to \mathbb{R}$ by

$$h(m, n) := \begin{cases} m \cdot s(\frac{1}{m}) & \text{if } m > 0 \text{ and } n \geq \frac{1}{s(\frac{1}{m})} \\ 0 & \text{otherwise} \end{cases}$$

If $m > 0$ and $n \geq \frac{1}{s(\frac{1}{m})}$, then $\frac{1}{n} \leq s^{-1}(\frac{1}{m})$ and since $s$ is increasing we get $h(m, n) = ms(\frac{1}{n}) \leq ms(s^{-1}(\frac{1}{m})) = \frac{1}{m}$. Because the set $U$ of all $(m, n)$ with $m > 0$ and $n \geq \frac{1}{s(\frac{1}{m})}$ together with $(0,0)$ is open in $X$, $h$ is continuous in $(0,0)$. Hence $h$ is continuous on $X$. For $(m, n) \in U$, $m > 0$ we have $n > 0$, $g(m, n) \cdot h(m, n) = \frac{1}{m} \cdot m \cdot s(\frac{1}{n}) = s(\frac{1}{n})$ and $(s \circ f)(m, n) = s(f(m, n)) = s(g(m, n)) = s(\frac{1}{n})$. Hence on $U$ we have $s \circ f = g \cdot h$ and (5.2)(i) implies that $g$ divides $s \circ f$. □

\[\text{------------------------------------------- LONG VERSION -------------------------------------------}\]

Recall that a pseudo compact space is a space $X$, such that every continuous function on $X$ is bounded (note that there are topological spaces which are pseudo compact, where each subset is a $G_δ$, which is not normal and not countably compact; cf. Gillman–Jerison, 5I; see also 5H. Also, a pseudo compact space which is real compact is compact, since in this case the isomorphism $C(βX) \to C(X)$ induces an homeomorphism $X \to βX$. More general, by [Gil-Jer], 8A, $X$ is pseudo compact if and only if $βX$ is the Hewitt real compactification of $X$.)

(9.7) \textbf{Definition.} A space $X$ is called \textbf{sequential} if for every $f \in C(X)$ and every open subset $U$ of $\{ f \neq 0 \}$ with $\overline{U} \cap \{ f = 0 \} \neq \emptyset$ there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $U$, bounded in $X$, such that $\lim_{n \to \infty} f(x_n) = 0$.

Observe that $(x_n)$ itself does not need to converge to anything.

(9.8) \textbf{Remark.} Let $I$ be a totally ordered set of indices. For $i, j \in I$ let $x_i \in \mathbb{R}$, $x_i > 0$ $(i \in I)$ such that $\lim_{i \in I} x_i = 0$. Then there is a countable sequence $i_1 < i_2 < \ldots$ in $I$ which is cofinal in $I$. In particular $\lim_{i \to \infty} x_{i_n} = 0$.

\textbf{Proof.} For $n \in \mathbb{N}$, there is some $i_n \in I$ such that $x_{i_n} < 1/n$ for all $i \in I$, $i_n \leq i$. If the set $\{ i_n \mid n \in \mathbb{N} \}$ is not cofinal in $I$, then there is some $i \in I$ with $i_n \leq i$ for all $n \in \mathbb{N}$. Then $x_i$, a contradiction to our assumption. □

Note that the remark above is not true anymore if we replace “totally ordered index set” by “filtered family”!

From (9.8) we get the following examples of sequential spaces:

If $X$ and $Y$ are sequential, is then also $X \times Y$ sequential ?

(9.9) \textbf{Definition.} Let $\mathfrak{a}$ be an ideal of $C(X)$. We say that $\mathfrak{a}$ is \textbf{algebraically simple} if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{f}} \cdot C(X)$ for some $f \in C(X)$.

Observe that $\mathfrak{a}$ is algebraically simple if $\sqrt{\mathfrak{a}} = \sqrt{f_1, \ldots, f_n}$ for some $f_1, \ldots, f_n \in C(X)$, since $\sqrt{f_1, \ldots, f_n} = \sqrt{f_1^2 + \ldots + f_n^2}$. \[\text{------------------------------------------- END OF LONG VERSION -------------------------------------------}\]
Observe that for any connected Tychonoff space $X$ and each $f \in C(X)$ with $\{0\} \not\subset \sqrt{f} \not\subset C(X)$ we have $(f) \not\subset O(f)$.

(9.10) Lemma. Let $f,g \in C^*(X)$ and let $q \in C(X)$ with $g = f \cdot q$. Then $f$ divides $q$ in $C^*(X)$ if and only if there is some $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ such that $q$ is bounded in $\{0 < |f| < \varepsilon\}$.

Proof. If $f$ divides $q$ in $C^*(X)$, then $q$ is bounded on $\{f \neq 0\}$. Conversely take $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ such that $q$ is bounded in $\{0 < |f| < \varepsilon\}$, say $|q| < N \in \mathbb{N}$ on $\{0 < |f| < \varepsilon\}$. Then $q$ is bounded by $K := \sup\{N, \frac{\varepsilon}{2} | |f(x)| \geq \varepsilon\}$ on $\{f \neq 0\}$ and $b := (q \wedge (K + 1)) \vee (-K - 1)$ is a bounded function with $g = f \cdot b$.

We conclude with consequences of (9.2).

(9.11) Corollary. If $X$ satisfies condition (★) of (9.2), and if $g \in C(X)$, then for every ideal $a$ of $C(X)$ with $\sqrt{a} = \sqrt{g}$ we have

$$(\sqrt{a})^\circ = a^\circ = a^\forall = (\sqrt{a})^\forall.$$ 

If in addition $X$ is normal, then $O(a) = a^\circ = a^\forall$.

Proof. The first assertion follows directly from (9.2). If $X$ is normal, then by (4.21) and (4.22) applied to the identities in (9.2) we get $O(a) = a^\circ = a^\forall$. □

It is not difficult to see that $O(g) = (g)^\circ$ for all $g \in C(X)$, where $X$ is the topological space considered in (9.6). The question if $O(g)$ can differ from $(g)^\circ$ in some normal space remains unanswered in this article.

(9.12) Corollary. Let $X$ be a normal space that satisfies condition (★) of (9.2). If $a, b$ are ideals of $C(X)$ such that $\sqrt{a}$ and $\sqrt{b}$ are simply generated as radical ideals, then

$$(a + b)^\circ = a^\circ + b^\circ.$$ 

Proof. By (4.8) and (9.11). □

Recall that by (3.13) the diamond-operation on prime ideals of $C(X)$ is additive and by (4.8), the $O$-operation on ideals of any real closed ring is additive. The following example shows that the diamond-operation on ideals of $C(X)$ is not additive:

(9.13) Example. Let $X \subseteq \mathbb{R}^n$ with $[-1,1]^n \subseteq X$, $n \geq 2$. Let $f(x) \in C(X)$ be defined by $f(x) := |x|$ and let $a := \sqrt{f}$. Then there is a prime ideal $p \in V(a^\circ)^{\text{min}}$ such that $p^\circ + a^\circ = p \not\subseteq (p + a)^\circ$.

Proof. There is a unique prime $z$-ideal $m$ of $C(X)$ containing $f$, namely $m = \hat{0}$, where $0$ denotes the origin in $\mathbb{R}^n$. Since $n \geq 2$, there is a proper specialization chain $p \hookrightarrow q \hookrightarrow m$ of prime $z$-ideals in $C(X)$ with $p \in (\text{Spec } C(X))^{\text{min}}$.

Since $p \hookrightarrow m \geq a$ we have $p \in O(V(a))$, hence $O(a) \subseteq p$. Since $X$ is normal and $X$ satisfies condition (★) of (9.2) we know $O(a) = a^\circ$. Therefore $p \in V(a^\circ)^{\text{min}}$. Since $p$ is $z$-radical we have $p^\circ + a^\circ = p + a^\circ = p$ and it remains to show $p \not\subseteq (p + a)^\circ$.

Since $q \neq m$, $q$ does not contain $a$, hence we can not have $p + a \subseteq q$. Since $q$ and $p + a$ are prime ideals containing $p$ we must have $p \subseteq q \subseteq p + a$. Since $q$ is $z$-radical we have $q \subseteq (p + a)^\circ$. Since $p \not\subseteq q$ we get $p \not\subseteq (p + a)^\circ$. □
The diamond-operation on ideals of $C(X)$ is not additive. Recall that by (3.13) the diamond-operation on prime ideals of $C(X)$ is additive and by (4.8), the $O$-operation on ideals of any real closed ring is additive.

(ii) The image of the map $V(a)^{\text{min}} \to V(a^\circ)$, $q \mapsto q^\circ$ is not contained in $V(a^{\circ})^{\text{min}}$.

**Proof.** There is a unique prime $z$-ideal $m$ of $C(X)$ containing $f$, namely $m = 0$, where 0 denotes the origin in $\mathbb{R}^n$. Since $n \geq 2$, there is a proper specialization chain $p \to q \to m$ of prime $z$-ideals in $C(X)$ with $p \in (\text{Spec } C(X))^{\text{min}}$. By entering in minimal points(ii), $p + a \in V(a)^{\text{min}}$.

Since $p \to m \ni a$ we have $p \in \partial(V(a))$, hence $O(a) \subseteq p$. Since $X$ is normal and $X$ satisfies condition (★) of (9.2) we know $O(a) = a^\circ$. Therefore $p \in V(a^{\circ})^{\text{min}}$. Since $p$ is $z$-radical we have $p^\circ + a^\circ = p + a^\circ = p$ and it remains to show $p \nsubseteq (p + a)^\circ$.

Since $q \neq m$, $q$ does not contain $a$, hence we cannot have $p + a \subseteq q$. Since $q$ and $p + a$ are prime ideals containing $p$ we must have $p \subseteq q \subseteq p + a$. Since $q$ is $z$-radical we have $q \subseteq (p + a)^\circ$. Since $p \nsubseteq q$ we get $p \nsubseteq (p + a)^\circ$. □

(9.15) REMARK. I conjecture that $O(f) = (f)^\circ$ for all $f \in C(X)$ and each Tychonoff space $X$. The notion “sequential space” is introduced to confirm this conjecture in many cases (cf. (9.4)). As $\beta X$ is sequential, one might ask if it possible to reduce the question $O(f) = (f)^\circ$ to $C^*(X)$. The problem with this approach is that algebraic simplicity is extremely destroyed when passing from $C(X)$ to $C^*(X)$: for example if $f \in C(\mathbb{R}^2)$ is one of the coordinate projections, then there is no $g \in C^*(X)$ with $g \in f \cdot C(X) \cap C^*(X) \subseteq \sqrt[n]{\mathbb{R}} C^*(X)$.

(9.16) EXAMPLE. Let $f \in C^*(X)$. Then in general we do not have $f \cdot C(X) \cap C^*(X) \subseteq f \cdot C^*(X)$. We even do not have $O(f \cdot C(X)) \cap C^*(X) \subseteq \sqrt[n]{f \cdot C^*(X)}$ in general. Note also that $O(f \cdot C(X)) \cap C^*(X) = O(f \cdot C(X) \cap C^*(X))$ by (4.26).

To see an example let $X = \mathbb{R}$, let $f \in C(\mathbb{R})$ with bounded zero set such that

$$\lim_{x \to \pm \infty} |f(x)| = 0.$$ 

Let $g \in C^*(X)$ such that $\{f = 0\}$ is in the interior of $\{g = 0\}$ and such that $g(x) \geq 1$ for $|x| >> 1$. Then $g \in O(f \cdot C(X)) \cap C^*(X)$. But $g \not\in \sqrt[n]{f \cdot C^*(X)}$, since $g \text{ mod } m \geq 1$ for every $m \in C^*(\mathbb{R})$ located at $\pm \infty$, whereas $f \in m$ for every $m \in C^*(\mathbb{R})$ located at $\pm \infty$. □

END OF LONG VERSION

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