ON IMPLICIT FUNCTION THEOREM
IN O-MINIMAL STRUCTURES

ZOFIA AMBROŻY AND WIESŁAW PAWŁUCKI

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Abstract. A local-global version of the implicit function theorem in o-minimal structures and a generalization of the theorem of Wilkie on covering open sets by open cells are proven.

1. Introduction. Assume that $R$ is any real closed field and an expansion of $R$ to some o-minimal structure is given. Throughout the paper we will be talking about definable sets and mappings referring to this o-minimal structure. (For fundamental definitions and results on o-minimal structures the reader is referred to [vdD] or [C].)

We will prove the following local-global version of implicit function theorem

Theorem. Let $p \in \{1, 2, \ldots\} \cup \{\infty, \omega\}$. Let $\Omega$ be a definable open subset of $R^{n+m}$ and let $F = (F_1, \ldots, F_m) : \Omega \rightarrow R^m$ be a definable $C^p$-mapping such that

$$\frac{\partial(F_1, \ldots, F_m)}{\partial(x_{n+1}, \ldots, x_{n+m})} \neq 0 \quad \text{in } \Omega.$$ 

Then there exists a finite family $f_i : C_i \rightarrow R^n$ ($i \in \{1, \ldots, s\}$) of definable $C^p$-mappings defined on definable open subsets $C_i$ of $R^n$ such that for each $i$

$$F(x_1, \ldots, x_n, f_i(x_1, \ldots, x_n)) = 0 \quad \text{on } C_i \quad \text{and} \quad \bigcup_{i=1}^{s} f_i = F^{-1}(0).$$

(We adopt the convention to identify mappings with their graphs.)

Remark 1. When $\Omega$ is bounded, it follows from a theorem of Wilkie [W] that in the above theorem each of $C_i$’s can be assumed to be an open definable cell in $R^n$.

By the classical implicit function theorem, the above theorem is an immediate corollary to the following

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**Proposition 1.** Let $M$ be a definable $C^p$-submanifold of $R^{n+m}$ of dimension $n$. Let $\pi : R^{n+m} = R^n \times R^m \rightarrow R^n$ be the natural projection. Assume that $\pi | M$ is a local diffeomorphism.

Then there exists a finite family $A_1, \ldots, A_s$ of open definable subsets of $M$ such that $M = A_1 \cup \cdots \cup A_s$ and, for each $i \in \{1, \ldots, s\}$, $\pi | A_i$ is a $C^p$-diffeomorphism (onto an open subset in $R^n$).

Proposition 1 is a consequence of the following, much more general, elementary fact.

**Proposition 2.** Let $E$ be any definable subset of $R^{n+m}$. Let $\pi : R^{n+m} = R^n \times R^{m} \rightarrow R^n$ be the natural projection. Assume that $\pi | E$ is locally injective.

Then there exists a finite family $A_1, \ldots, A_s$ of open definable subsets of $E$ such that $E = A_1 \cup \cdots \cup A_s$ and, for each $i \in \{1, \ldots, s\}$, $\pi | A_i$ is injective.

As an application of Proposition 1, we will prove the following generalization of the mentioned above theorem of Wilkie.

**Proposition 3.** Any definable bounded $C^p$-submanifold $M$ of $R^{n+m}$ of dimension $n$ can be represented as a finite union $M = C_1 \cup \cdots \cup C_s$, where each of $C_i$’s is, after perhaps a permutation of coordinates in $R^{n+m}$, a definable $n$-dimensional cell in $R^{n+m}$.

2. **Proof of Proposition 2.** We will prove by induction on $k \in \{0, \ldots, n\}$ that if $C$ is a definable subset of $E$ of dimension $k$, then there exists a finite family $A_1, \ldots, A_s$ of open definable subsets of $E$ such that $C \subset A_1 \cup \cdots \cup A_s$ and each of $\pi | A_i$ ($i \in \{1, \ldots, s\}$) is injective.

By using a cell decomposition we can assume without any loss of generality that $C$ is a $k$-dimensional cell. Then $\pi | C$ is injective and $\pi (C)$ is a $k$-dimensional cell in $R^n$. After perhaps a permutation of coordinates in $R^n$, one can assume that

$$\pi (C) = \{(x_1, \ldots, x_n) : (x_1, \ldots, x_k) \in \Omega, x_j = \varphi_j (x_1, \ldots, x_k) (j = k + 1, \ldots, n)\}$$

and

$$C = \{(x_1, \ldots, x_{n+m}) : (x_1, \ldots, x_k) \in \Omega, x_j = \varphi_j (x_1, \ldots, x_k) (j = k+1, \ldots, n+m)\},$$

where $\Omega$ is definable open subset of $R^k$ and $\varphi_j : \Omega \rightarrow R$ ($j = k + 1, \ldots, n + m$) are definable continuous functions.

For each $u = (u_1, \ldots, u_k) \in \Omega$ and $\varepsilon > 0$, set

$$\Theta (u, \varepsilon) := \{(x_1, \ldots, x_{n+m}) \in E : u = (x_1, \ldots, x_k), | x_j - \varphi_j (u) | < \varepsilon (j = k + 1, \ldots, n + m)\}. $$

For each $u \in \Omega$, set $r (u) := \sup \{ \varepsilon \in (0,1] : \pi | \Theta (u, \varepsilon)$ is injective$\}$. By the assumption of local injectivity $r$ is well-defined and it is easy to check that $r$ is definable. There exists a closed definable subset $Z$ of $\Omega$ of dimension $< k$ such that $r | \Omega \setminus Z$ is continuous. It is clear that $\pi$ is injective in restriction to the set

$$\bigcup_{u \in \Omega \setminus Z} \Theta (u, r (u)) = \{(x_1, \ldots, x_{n+m}) \in E : u = (x_1, \ldots, x_k) \in \Omega \setminus Z, | x_j - \varphi_j (u) | < r (u) (j = k + 1, \ldots, n + m)\},$$

which is an open definable neighborhood of $C | \Omega \setminus Z$ in $E$. Now, to finish the proof it suffices to apply the induction hypothesis to $C | Z$. 
3. Proof of Proposition 3. Let \( \tau : M \ni x \mapsto T_x M \in \mathbb{G}_n(\mathbb{R}^{n+m}) \) be the Gauss map, \( \{e_j\} \ (j \in \{1, \ldots, n + m\}) \) - the canonical basis of \( \mathbb{R}^{n+m} \) and let

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V_\alpha := \{ L \in \mathbb{G}_n(\mathbb{R}^{n+m}) : L \cap \left( \sum_{i=1}^{k} R e_{\alpha_i} \right) = \{0\} \},
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_k), \ 1 \leq \alpha_1 < \cdots < \alpha_k \leq n + m \). Since \( \tau \) is definable, \( \{\tau^{-1}(V_\alpha)\} \) is an open definable covering of \( M \). This reduces the general case, after perhaps a permutation of coordinates, to that from Proposition 1. Hence, Proposition 3 follows from Proposition 1 and Remark 1.

Remark 2. It seems that the assumption in Proposition 3 that \( M \) is bounded is superfluous.

References


Zofia Ambroży; Instytut Matematyczny Polskiej Akademii Nauk, ul. Śniadeckich 8, 00-656 Warszawa  
e-mail: zambrozy@gmail.com

Wiesław Pawłucki; Instytut Matematyki Uniwersytetu Jagiellońskiego, ul. Prof. St. Łojasiewicza 6, 30-348 Kraków, Poland  
e-mail: Wieslaw.Pawlucki@im.uj.edu.pl