# GROMOV-HAUSDORFF LIMITS IN DEFINABLE FAMILIES

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Abstract. The notion of piecewise definable metric space is introduced. It is shown that the set of all Gromov-Hausdorff limits of piecewise definable spaces belonging to a fixed bounded definable family is again a definable family. The Gromov-Haudorff limit is taken with respect to the geodesic metric and the word *definable* means *definable in some o-minimal structure over* $\mathbb{R}$.

## 1. Introduction and statement of results

A piecewise definable set is, roughly speaking, a metric space obtained by glueing finitely many definable sets along definable sets. Here the word *definable* means *definable in some o-minimal structure over the reals*. See [13] and [14] for o-minimal structures and the next section for the definition of piecewise definable sets.

One can associate to a compact piecewise definable set in a canonical way a geodesic metric. The aim of this note is to study Gromov-Hausdorff limits of piecewise definable sets belonging to a fixed definable family.

Let $A \subset \mathbb{R}^{m+n}$ be a bounded piecewise definable set. Let $A' := \pi_m(A)$, where $\pi_m : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is the projection onto the first coordinates. Each fiber of $\pi_m$ over a point $a \in A'$ can be considered as a piecewise definable space in $\mathbb{R}^n$, which we suppose to be compact and which we denote by $A_a$. Let $F(A) := \{(A_a, d_{A_a}) : a \in A'\}$ denote the set of geodesic metric spaces in the family $A$.

For any family $F$ of compact metric spaces, we denote by $cl(F)$ the closure of $F$ in the Gromov-Hausdorff topology, i.e. the family of all compact metric spaces which are Gromov-Hausdorff limits of sequences in $F$. We say that $F$ is definable if there exists a bounded definable family $A$ of piecewise definable compact sets as above with $F = F(A)$.

**Theorem 1.** *Let $F$ be a definable family of compact metric spaces. Then $cl(F)$ is also a definable family of compact metric spaces.*

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**Corollary 1.1.** *Let $X$ be the Gromov-Hausdorff limit of a sequence $X_1, X_2, \ldots$ of compact piecewise definable sets belonging to a fixed bounded definable family. Then $X$ is piecewise definable. Suppose that also the Hausdorff limit $Y$ of this sequence exists. Then there exists a finite-to-one map $\pi : X \to Y$ which preserves lengths of curves. The number of points in each fiber is bounded by a constant which depends only on the family.*

As an example, consider a family of ellipsoids in $\mathbb{R}^3$ getting flatter and flatter. As Gromov-Hausdorff limit, we obtain a double disc, which is clearly piecewise definable.

The analogon of Theorem 1, but with Gromov-Hausdorff limit replaced by Hausdorff limit, and geodesic metric replaced by Euclidean metric, is well-known. In the semialgebraic setting, this goes back to Bröcker ([4]) and was later extended, using model theory, to o-minimal structures by Marker-Steinhorn ([9]), Pillay ([11]) and van den Dries ([12]). Lion-Speissegger gave a geometric proof of the same fact ([8]), and their version will be used in the proof of our main theorem. Gromov-Hausdorff limits, but still with respect to Euclidean metric, were considered by van den Dries ([12]).

For geometric and practical applications, the geodesic metric is more interesting and more natural than the Euclidean one. However, it is much less understood. One obstacle when dealing with the geodesic metric is that, in general, it is not a definable function. In [1] it is shown that the Gromov-Hausdorff limit of a definable 1-parameter family exists; and this fact was used to study the local geometry of definable sets. Our main theorem extends this result in two directions: first we allow arbitrary definable families and secondly we describe all limit spaces as piecewise definable sets.

## 2. Piecewise definable metric spaces

Let $X^1, \ldots, X^k, X^{12}, X^{13}, \ldots, X^{k-1,k} \subset \mathbb{R}^n$ be compact definable subsets with $X^{ij} \subset X^i \cap X^j$.

We extend the geodesic metrics $d_{X^j}$ on $X^j$ to a metric $d$ on the disjoint union $\sqcup_{j=1}^k X^j$ by setting $d(x,y) = d_{X^j}(x,y)$ if $x$ and $y$ are both contained in some $X^j$ and $d(x,y) = \infty$ otherwise. We consider the equivalence relation generated by the $X^{ij}$. The quotient pseudo metric space is denoted by $X := (\sqcup_{j=1}^k X^j, d_X)$ (see [3] for quotients of metric spaces) and called a *piecewise definable metric space*.

Let us describe $d_X$ more explicitly. For $x, y \in X$ we have

$$d_X(x,y) = \inf \left\{ \sum_{j=1}^{N} d(x^j, y^j) \right\}$$

where the infimum runs over all finite sequences $x^1 = x, y^1, x^2, y^2, \dots, x^N, y^N = y$ such that $y^j$ and $x^{j+1}$ are equivalent for $j = 1, \dots, N-1$ (there is no bound on $N$).

**Lemma 2.1.** $(X, d_X)$ *is a proper length space. In particular, it is a geodesic space.*

*Proof.* By definition, $d_X$ is a pseudo-metric. We first have to show that $d_X(x,y) = 0$ implies that $x$ and $y$ are equivalent.

Let $\pi : X \to \cup_{j=1}^k X^j \subset \mathbb{R}^n$ be the canonical projection map. Then for all $x, y \in X$ we get $d_X(x,y) \geq \|\pi(x) - \pi(y)\|$.

Let $r$ be the minimal Euclidean distance between $\pi(x)$ and one of those $X^{ij}$ which do not contain $\pi(x)$. Denote the canonical embedding of $X^j$ in $X$ by $\tau_j$.

Choose a sequence $x^1 = x, y^1, x^2, y^2, \dots, x^N, y^N = y$ as above with $\sum_{j=1}^N d(x^j, y^j) < r$. Let $i_j$ be such that $x^j, y^j \in X^{i_j}$. Since $y^j$ and $x^{j+1}$ are equivalent and $\|\pi(y^j) - \pi(x)\| < r$, we get by definition of $r$ $\tau_{i_j}(\pi(x)) \sim \tau_{i_{j+1}}(\pi(x))$. This holds for all $j$ and shows that $x = \tau_{i_1}(\pi(x)) \sim y = \tau_{i_N}(\pi(y))$. Therefore $d_X$ is a metric.

Given a sequence $x = x^1, y^1, \dots, x^N, y^N$, we can join $x^j$ and $y^j$ by a geodesic in $X^{i_j}$. Pasting these curves together yields a continuous curve between $x$ and $y$ whose length is $\sum_{j=1}^N d(x^j, y^j)$. This implies that $d_X$ is a length metric.

Since $(X, d_X)$ is complete and locally compact, it is proper and, by Hopf-Rinow, a geodesic metric space ([3]). $\square$

## 3. Proof of the main theorem modulo some propositions

**Definition 3.1.** *Let a piecewise definable space $X$ be given by sets $X^1, \dots, X^N$ and glueing sets $X^{12}, \dots, X^{N-1,N}$. A subdivision is a piecewise definable set $Y$ given by sets $Y^1, \dots, Y^M$ and glueing sets $Y^{12}, \dots, Y^{M-1,M}$ such that*

a) *each $X^i$ is a union of some of the $Y^j$,*
b) *if $Y^{j_1}, Y^{j_2} \subset X^i, j_1 \neq j_2$, then $Y^{j_1 j_2} = Y^{j_1} \cap Y^{j_2}$,*
c) *if $Y^{j_1} \subset X^{i_1}, Y^{j_2} \subset X^{i_2}$ with $i_1 \neq i_2$ then $Y^{j_1 j_2} = Y^{j_1} \cap Y^{j_2} \cap X^{i_1 i_2}$.*

**Lemma 3.2. (Subdivision lemma)**
*If $Y$ is a subdivision of $X$, then $d_Y = d_X$.*

**Definition 3.3.** *A compact definable set $X \subset \mathbb{R}^n$ is called $C$-normal, where $C > 1$ is a real number, if $d_X(x,y) \leq C\|x - y\|$ for all $x, y \in X$.*

If $X$ is $C$-normal for some $C > 1$, then $X$ is also called normally embedded (cf. [2]).

**Proposition 3.4. (Convergence of normal sets)**
*Let $X_1, X_2, \dots$ belong to a fixed bounded definable family of compact subsets*

*of $\mathbb{R}^n$. Suppose that each $X_i$ is $C$-normal for some fixed constant $C > 1$, and that the Hausdorff limit $X := \lim_{i\to\infty} X_i$ exists. Let $x_i, y_i \in X_i$ and suppose $x_i \to x, y_i \to y$ for $i \to \infty$. Then*

$$d_X(x,y) = \lim_{i\to\infty} d_{X_i}(x_i, y_i).$$

*In particular, $X$ is also $C$-normal.*

Note that the assumption on $X_i$ can not be dropped, as can be seen for the example of flat ellipsoids in $\mathbb{R}^3$.

**Proposition 3.5. (Convergence of normal families)** *Let $X_1, X_2, \ldots$ belong to a fixed bounded definable family of compact piecewise definable sets, such that $X_i$ is given by sets $X_i^1, \ldots, X_i^k \subset \mathbb{R}^n$ and $X_i^{1,2}, \ldots, X_i^{k-1,k} \subset \mathbb{R}^n$. Suppose that $X_i^1, \ldots, X_i^k$ are $C$-normal for some $C > 1$ independent of $i$. Suppose furthermore that each Hausdorff limit $X^j := \lim_{i\to\infty} X_i^j, X^{j,l} := \lim_{i\to\infty} X_i^{j,l}$ exists. Then the piecewise definable space $X$ given by the sets $X^1, \ldots, X^k, X^{1,2}, \ldots, X^{k-1,k}$ is the Gromov-Hausdorff limit of the sequence $(X_i, d_{X_i})$ (and $\pi(X)$ is the Hausdorff limit of this sequence).*

We postpone the proofs of the preceding propositions to later sections.

*Proof of the Theorem 1.* Let $A \subset \mathbb{R}^{m+n}$ be bounded and piecewise definable, $A' = \pi_m(A)$ and $A_a, a \in A'$ (which is supposed to be compact) the piecewise definable space canonically associated to a fiber.

Let us suppose that $A$ is given by the pieces $A^1, \ldots, A^N, A^{12}, \ldots, A^{N-1,N}$. By taking a subdivision if necessary, we can suppose that $A_a^1, \ldots, A_a^N$ are $C$-normal for some constant $C > 1$ and all $a \in A'$. This follows from the theorem of Kurdyka-Orro ([7]). The set $F(A)$ remains the same by Lemma 3.2.

Consider the family $\tilde{A} \subset A' \times \mathbb{R}^{n\left(N+\binom{N}{2}\right)}$ with $\tilde{A}_a = A_a^1 \times \ldots \times A_a^N \times A_a^{12} \times \ldots \times A_a^{N-1,N} \subset \mathbb{R}^{n\left(N+\binom{N}{2}\right)}$. Applying the theorem of Lion-Speissegger ([8]) to this family and noting that Hausdorff convergence of a product is equivalent to Hausdorff convergence of each of its factors, we get an integer $M$ and a compact, definable family $B \subset \mathbb{R}^{M+n}$ of piecewise definable sets such that

a) for every $a \in A'$ there exists $b \in B' = \pi_M(B)$ with $A_a = B_b$;
b) for every sequence $(b_i)_i$ in $B'$ such that $\lim_{i\to\infty} b_i = b$, each piece of $B_{b_i}$ converges in the Hausdorff topology to the corresponding piece of $B_b$.

Note that b) implies, by Propositions 3.4 and 3.5, that $(B_{b_i}, d_{B_{b_i}})$ converge in the Gromov-Hausdorff metric to $(B_b, d_{B_b})$.

Replacing $B'$ by the closure of the definable set $\{b \in B' : \exists a \in A' : A_a = B_b\}$ and $B$ by $\pi_M^{-1}(B') \cap B$, we get a compact, definable family which satisfies a) and b) and moreover

c) The set of $b \in B'$ such that there exists $a \in A'$ with $A_a = B_b$ is dense in $B'$.

We claim that $cl(F(A)) = F(B)$.

The compactness of $B$ and d) imply that $cl(F(B)) = F(B)$. Since $F(A) \subset F(B)$, we get $cl(F(A)) \subset F(B)$.

For $b \in B'$, choose a sequence $b_i \in B'$ converging to $b$ such that there exist $a_i \in A'$ with $A_{a_i} = B_{b_i}$. This is possible by c). The metric spaces $A_{a_i} = B_{b_i}$ converge to $B_b$ by b). It follows $B_b \in cl(F(A))$ and thus $F(B) \subset cl(F(A))$.

This finishes the proof of Theorem 1 (modulo the proof of the propositions, which will be given in the next sections). □

*Proof of Corollary 1.1.* Let $A$ be a bounded definable family of compact piecewise definable spaces such that for each $i$, there exists $a_i \in A'$ with $X_i = A_{a_i}$. Let $B$ be a compact definable family as in the proof of the theorem and $b_i \in B'$ with $X_i = B_{b_i}$. Passing to a subsequence if necessary, we can suppose that $b_i \to b'$ for some $b' \in B'$. Then Property b) of $B$ and Proposition 3.5 imply that $\pi(X) = \pi(B_b)$ is the Hausdorff limit of the sequence. The cardinality of $\pi^{-1}(x'), x' \in \pi(X)$ is bounded by the number $N$ of pieces in a $C$-normal subdivision of $B$.

The length of a curve $\gamma$ in $X$ is - almost by definition - the same as the length of the curve $\pi \circ \gamma$ in $\pi(X)$. □

## 4. Proof of the Subdivision lemma

*Proof.* Let $Y$ denote a subdivision of $X$. As sets, $X = Y$, we have to show that $d_Y = d_X$.

By definition,

$$d_X(x, y) = \inf \left\{ \sum_{j=1}^{N} d(x^j, y^j) \right\}$$

where the infimum runs over all finite sequences $x^1 = x, y^1, x^2, y^2, \ldots, x^N, y^N = y$ such that $y^j$ and $x^{j+1}$ are equivalent (with no bound on $N$) and similarly for $Y$.

Given any sequence of points as above for $d_Y$, we delete all pairs $y^j, x^{j+1}$ which lie in the same $X^j$. This gives a sequence as in the definition for $d_X$, whose length is not longer by triangle inequality and the fact that $d_{Y^j} \geq d_{X^i}|_{Y^j}$ for $Y^j \subset X^i$. We deduce that $d_X \leq d_Y$.

Conversely, given a sequence $x^1, y^1, \ldots, x^N, y^N$ for $d_X$ and $\epsilon > 0$, we can join each pair $x^j, y^j \in X^{\mu_j}$ by a definable curve in $X^{\mu_j}$ of length less than $d_{X^{\mu_j}}(x^j, y^j) + \epsilon$. Since the curve is definable, we can partition it into finitely many parts which are completely contained in one of the $Y^j$. Then we take the endpoints of these parts as new points and obtain a sequence as in the definition for $d_Y$. The length of this new sequence is bounded by the length of the old one plus $\epsilon N$. It follows $d_Y \leq d_X + \epsilon N$. Since $\epsilon$ was arbitrary, we get $d_Y \leq d_X$. □

## 5. Convergence of normal sets

**Lemma 5.1.** *Let $X \subset \mathbb{R}^n$ be a connected, compact, definable set. Then there exists a continuous monoton definable function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $d_X(x,y) \leq \phi(\|x-y\|)$.*

*Proof.* By the theorem of Kurdyka-Orro ([7]), there exists a continuous definable function $d_{def} : X \times X \to \mathbb{R}$ with $d_X \leq d_{def} \leq 2d_X$.

Recall that the Łojasiewicz inequality in o-minimal structures states that if $f, g$ are continuous definable functions on a compact definable set with $f^{-1}(0) \subset g^{-1}(0)$, then there exists a monotone definable function $\phi$ with $g \leq \phi \circ f$ and $\phi(0) = 0$. I did not find a precise reference for this inequality, except the unpublished [6], but the proof is very straightforward.

The statement of the lemma follows immediately by applying this with $g := d_{def} : X \times X \to [0,\infty)$ and $f$ equal to the Euclidean distance $X \times X \to [0,\infty)$. □

**Definition 5.2.** *Let $X \subset \mathbb{R}^n$ be a connected compact definable set and $x, y \in X$. An $\epsilon$-path $c$ between $x$ and $y$ is a sequence $c = (x_1, x_2, \ldots, x_N)$ of points of $X$ such that $x_1 = x, x_N = y$, $\|x_{i+1} - x_i\| \leq \epsilon$. The length of $c$ is given by $l(c) := \sum_{i=1}^{N-1} \|x_{i+1} - x_i\|$.*

**Lemma 5.3.** *Let $X \subset \mathbb{R}^n$ be a compact definable set and $x, y \in X$. Define*

$$d_X^\epsilon(x,y) := \inf\left\{l(c) : c \text{ is an } \epsilon\text{-path between } x, y\right\}.$$

*Then $\lim_{\epsilon\to 0} d_X^\epsilon(x,y) = d_X(x,y)$.*

*Proof.* Let $\gamma$ be a geodesic between $x$ and $y$. Choosing points on $\gamma$ at distances $\leq \epsilon$, we get that $d_X^\epsilon(x,y) \leq d(x,y)$, therefore $\limsup_{\epsilon\to 0} d_X^\epsilon(x,y) \leq d(x,y)$.

For the opposite direction, fix $\eta > 1$ and a covering $X = \cup_{j=1}^k X^j$ by compact $\eta$-normal subsets $X^j$ ([7]).

Given a sequence $x_1 = x, x_2, \ldots, x_N = y$ with $\|x_{i+1} - x_i\| \leq \epsilon$, we construct a new sequence of this type as follows.

Let $n_0 := 0$ and let $j_1 \in \{1, \ldots, k\}$ be such that $x_1 \in X^{j_1}$. Let $n_1$ be the largest integer (possibly equal to 1) such that $x_{n_1} \in X^{j_1}$.

If $n_1 < N$, let $j_2$ be such that $x_{n_1+1} \in X^{j_2}$. Let $n_2$ be the largest integer with $x_{n_2} \in X^{j_2}$.

We continue in this way. After $k' \leq k$ steps, the process finishes and we get finite sequences $j_1, j_2, \ldots, j_{k'}$ and $0 = n_0 < n_1 < \ldots < n_{k'} = N$ such that $x_{n_i+1}$ and $x_{n_{i+1}}$ belong to $X^{j_{i+1}}$ for $i = 0, \ldots, k'-1$.

Choose for each $i = 0, \ldots, k'-1$ a sequence of points $z_1^i = x_{n_i+1}, z_2^i, \ldots, z_{N_i}^i = x_{n_{i+1}}$ in $X^{j_{i+1}}$ such that $d_{X^{j_{i+1}}}(z_j^i, z_{j+1}^i) \leq \epsilon$ and $\sum_{j=1}^{N_i-1} d_{X^{j_{i+1}}}(z_j^i, z_{j+1}^i) =$

$d_{X^{j_{i+1}}}(x_{n_i+1}, x_{n_{i+1}})$. The existence of such points follows as above by subdividing a geodesic joining $x_{n_i+1}$ and $x_{n_{i+1}}$ in $X^{j_{i+1}}$.

The sequence $z_1^0 = x, \ldots, z_{N_0}^0 = x_{n_1}, z_1^1 = x_{n_1+1}, \ldots, z_{N_{k'}}^{k'} = y$ still has the property that consecutive terms are at Euclidean distance at most $\epsilon$. From

$$\sum_{j=n_{i+1}}^{n_{i+1}-1} \|x_{j+1} - x_j\| \geq \|x_{n_{i+1}} - x_{n_i+1}\| \geq \eta^{-1} d_{X^{j_{i+1}}}(x_{n_i+1}, x_{n_{i+1}}) \geq \eta^{-1} \sum_{j=1}^{N_i-1} \|z_j^i - z_{j+1}^i\|$$

we see that the length of the new sequence is at most $\eta$ times the length of the original sequence.

Let $\phi$ be a function as in Lemma 5.1. Then

$$\begin{aligned} d_X(x,y) &\leq \sum_{i=0}^{k'} \sum_{j=1}^{N_i-1} d_{X^{j_{i+1}}}(z_{j+1}^i, z_j^i) + \sum_{i=1}^{k'-1} d_X(x_{n_i}, x_{n_i+1}) \\ &\leq \eta \sum_{i=0}^{k'} \sum_{j=1}^{N_i-1} \|z_{j+1}^i - z_j^i\| + k\phi(\epsilon) \\ &\leq \eta^2 \sum_{i=1}^{N-1} \|x_i - x_{i+1}\| + k\phi(\epsilon). \end{aligned}$$

It follows that $d_X(x,y) \leq \eta^2 d_X^\epsilon(x,y) + k\phi(\epsilon)$. Letting $\epsilon$ tend to 0 we obtain $d_X(x,y) \leq \eta^2 \liminf_{\epsilon\to 0} d_X^\epsilon(x,y)$. Since $\eta > 1$ was arbitrary, we even have $d_X(x,y) \leq \liminf_{\epsilon\to 0} d_X^\epsilon(x,y)$ and the lemma is proved. □

*Proof of Proposition 3.4.* Proceeding as in the previous proof, but with the explicit choice $\phi(t) = Ct$, we get for each $i$, all $\epsilon > 0$ and $\eta > 1$

$$d_{X_i}(x_i, y_i) \leq \eta^2 d_{X_i}^\epsilon(x_i, y_i) + C(\eta)\epsilon,$$

where $C(\eta)$ only depends on $\eta$, but not on $i$.

Fix $\epsilon > 0$ and $\eta > 1$. Choose an $\epsilon$-path $c$ in $X$ between $x$ and $y$ of length $l(c) \leq d_X^\epsilon(x,y) + \epsilon$. Since $X$ is the Hausdorff limit of $X_1, X_2, \ldots$, we find a sequence of $2\epsilon$-paths $c_i$ in $X_i$ between $x_i$ and $y_i$ converging to $c$. Triangle inequality implies $l(c) = \lim_{i\to\infty} l(c_i)$. On the other hand, $l(c_i) \geq d_{X_i}^{2\epsilon}(x_i, y_i) \geq \eta^{-2}(d_{X_i}(x_i, y_i) - 2C(\eta)\epsilon)$.

We deduce that

$$d_X^\epsilon(x,y) \geq l(c) - \epsilon \geq \eta^{-2}\left(\limsup_{i\to\infty} d_{X_i}(x_i, y_i) - 2C(\eta)\epsilon\right) - \epsilon.$$

Letting $\epsilon$ tend to 0 and afterwards $\eta$ tend to 1 we obtain $d_X(x,y) \geq \limsup_{i\to\infty} d_{X_i}(x_i, y_i)$.

Now let us prove the other direction.

Since the $C$-normal sets $X_1, X_2, \dots$ belong to a fixed bounded definable family, their geodesic diameters are uniformly bounded.

Let $\gamma_i$ be a geodesic in $X_i$ between $x_i$ and $y_i$. Given $\epsilon > 0$, we can choose sufficiently many points on $\gamma_i$ in order to get an $\epsilon$-path between $x_i$ and $y_i$ whose length is not larger than $d_{X_i}(x_i, y_i)$. Actually, since the length of $\gamma_i$ is uniformly bounded, the number of points needed is bounded from above by some number $N(\epsilon)$ which is independent of $i$.

Passing to a subsequence if necessary, we can assume that these $\epsilon$-paths converge to an $\epsilon$-path between $x$ and $y$. Triangle inequality implies that its length is bounded by $\liminf_{i\to\infty} d_{X_i}(x_i, y_i)$. Therefore $d_X^\epsilon(x, y) \le \liminf_{i\to\infty} d_{X_i}(x_i, y_i)$. Taking the limit as $\epsilon$ tends to 0 yields $d_X(x, y) \le \liminf_{i\to\infty} d_{X_i}(x_i, y_i)$. □

## 6. Convergence of normal families

**Lemma 6.1.** *In the situation of Proposition 3.5, suppose that $x_i \in X_i^{\mu_x}$ converges to $x \in X^{\mu_x}$ and that $y_i \in X_i^{\mu_y}$ converges to $y \in X^{\mu_y}$ as $i \to \infty$. Then*

$$d_X(x, y) = \lim_{i\to\infty} d_{X_i}(x_i, y_i).$$

*Proof.* Choose $\epsilon > 0$ and a sequence $x = x^1, y^1, \dots, x^N, y^N = y$ such that $x^j, y^j$ belong to the same $X^{\mu_j}$, $y^j$ and $x^{j+1}$ belong to $X^{\mu_j,\mu_{j+1}}$ and such that

$$\sum_{j=1}^N d_{X^{\mu_j}}(x^j, y^j) \le d_X(x, y) + \epsilon.$$

We can choose a sequence $x_i^1 = x_i, y_i^1, x_i^2, y_i^2, \dots, x_i^N, y_i^N = y_i$ in each $X_i$ such that $x_i^j, y_i^j$ belong to $X_i^{\mu_j}$; $y_i^j$ and $x_i^{j+1}$ belong to $X_i^{\mu_j,\mu_{j+1}}$ and such that $x_i^j \to x^j, y_i^j \to y^j$ for $i \to \infty$.

By Proposition 3.4, the distances $d_{X_i^{\mu_j}}(x_i^j, y_i^j)$ converge to $d_{X^{\mu_j}}(x^j, y^j)$ as $i \to \infty$. Since $\sum_{j=1}^N d_{X_i^{\mu_j}}(x_i^j, y_i^j) \ge d_{X_i}(x_i, y_i)$ we obtain

$$\begin{aligned} d_X(x, y) \ge \sum_{j=1}^N d_{X^{\mu_j}}(x^j, y^j) - \epsilon &= \lim_{i\to\infty} \sum_{j=1}^N d_{X_i^{\mu_j}}(x_i^j, y_i^j) - \epsilon \\ &\ge \limsup_{i\to\infty} d_{X_i}(x_i, y_i) - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we thus have $d_X(x, y) \ge \limsup_{i\to\infty} d_{X_i}(x_i, y_i)$.

For the other direction, fix $\eta > 1$. By subdividing if necessary, we can assume that each $X_i^j$ is $\eta$-normal.

Define

$$\tilde{d}_{X_i}(x, y) = \inf \left\{ \sum_{j=1}^N \|x_i^j - y_i^j\| \right\}$$

where the infimum runs over all finite sequences $x_i^1 = x_i, y_i^1, x_i^2, y_i^2, \ldots, x_i^N, y_i^N = y$ such that $x_i^j$ and $y_i^j$ lie in the same $X_i^{\mu_j}$ and $y_i^j$ and $x_i^{j+1}$ lie in $X^{\mu_j,\mu_{j+1}}$.

Clearly $\tilde{d}_{X_i} \leq d_{X_i} \leq \eta \tilde{d}_{X_i}$. Working with $\tilde{d}_{X_i}$ has the advantage that we can use a uniform bound on the number $N$, namely the number of sets in the description of $X_i$ as piecewise definable space. This follows at once from triangle inequality for the Euclidean distance. We also get that the infimum is a minimum.

Choose for each $i$ a minimal sequence $x_i = x_i^1, y_i^1, \ldots, x_i^N, y_i^N = y_i$ as above. By passing to a subsequence, we can assume that $x_i^j \to x^j, y_i^j \to y^j$ for $i \to \infty$. Then $x^1 = x, y^N = y$, $x^j, y^j \in X^{\mu_j}$, $y^j, x^{j+1} \in X^{\mu_j,\mu_{j+1}}$.

By Proposition 3.4 we get that $X^{\mu_j}$ is $\eta$-normal and therefore

$$d_X(x,y) \leq \sum_{j=1}^N d_{X^{\mu_j}}(x^j, y^j) \leq \eta \sum_{j=1}^N \|x^j - y^j\| = \eta \lim_{i\to\infty} \sum_{j=1}^N \|x_i^j - y_i^j\|$$
$$= \eta \lim_{i\to\infty} \tilde{d}_{X_i}(x_i, y_i) \leq \eta \liminf_{i\to\infty} d_{X_i}(x_i, y_i).$$

This is true for every $\eta > 1$, hence $d_X(x,y) \leq \liminf_{i\to\infty} d_{X_i}(x_i, y_i)$. □

*Proof of Proposition 3.5.* Fix $\epsilon > 0$ and a finite $\epsilon$-dense net $\{x^1, \ldots, x^k\}$ in $(X, d_X)$. Since $X$ is piecewise definable of bounded geodesic diameter, the existence of such a net is clear.

Each point $x^j$ of this net lies in (at least) one of the sets $X^1, \ldots, X^k$, say $X^{\mu_j}$. We choose a sequence of points $x_i^j \in X_i^{\mu_j}$ converging to $x^j$.

By Lemma 6.1, $\lim_{i\to\infty} d_{X_i}(x_i^{j_1}, x_i^{j_2}) = d_X(x^{j_1}, x^{j_2})$.

We claim that $\{x_i^1, \ldots, x_i^k\}$ is a $2\epsilon$-net in $X_i$ for $i$ sufficiently large. If not, we could find a sequence of points $p_i \in X_i$ with distance to these sets at least $2\epsilon$. By passing to subsequences, we can assume that $p_i \in X_i^\mu$ for some fixed $\mu$ and that $p_i$ converges to some $p \in X^\mu$. Then the distance from $p$ to $\{x^1, \ldots, x^k\}$ is at least $2\epsilon$ (by Lemma 6.1), contradiction.

Since the Gromov-Hausdorff distance between a metric space and a finite $\epsilon$-net in it is bounded by $\epsilon$, we get, using triangle inequality for Gromov-Hausdorff distance, that $(X_i, d_{X_i})$ converges in the Gromov-Hausdorff distance to $(X, d_X)$.

The fact that $\pi(X)$ is the Hausdorff limit of the sequence $X_1, X_2, \ldots$ is trivial. □

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