O-minimal cohomology with definably compact supports

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June 16, 2004

Abstract

We define here the o-minimal cohomology theory with definably compact supports and prove that this theory is invariant in elementary extensions, in o-minimal expansions and coincides with its topological analogue for o-minimal structures in the real numbers. As an application we show that on definably locally compact definable sets the o-minimal Euler characteristic coincides with the Euler-Poincaré characteristic with definably compact supports.

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*With partial support from the FCT (Fundação para a Ciência e Tecnologia), program POCTI (Portugal/FEDER-EU).

1 Introduction

We work over an o-minimal expansion $\mathcal{N} = (N, 0, 1, <, +, \cdot, \dots)$ of a real closed field. Definable means $\mathcal{N}$-definable (possibly with parameters). As it is known, o-minimality is a wide ranging generalization of semi-algebraic geometry.

In the semi-algebraic case, Delfs constructs in [D] the semi-algebraic sheaf cohomology with definably compact supports. Semi-algebraic simplicial and singular (co)homology were constructed by Delfs and Knebusch in [dk1] based on the semi-algebraic sheaf cohomology.

For o-minimal expansions of real closed fields, Woerheide gives a direct construction of the o-minimal simplicial and singular homology with coefficients in $\mathbb{Z}$ in [Wo]. This construction easily gives, as in the classical case treated in [d], the o-minimal simplicial and singular homology and cohomology with arbitrary constant coefficients (see [ew]).

Woerheide's results are based on the definable triangulation theorem and on the method of acyclic models from homological algebra and are rather complicated due to the fact that, in arbitrary o-minimal expansions of fields, the classical simplicial approximation theorem and the method of repeated barycentric subdivisions and the Lebesgue number property for the standard simpleces Δ^n fail.

The goal of this paper is to construct in a simple and direct way the o-minimal cohomology (H_c^*, d_c^*) with definably compact supports and constant coefficients starting from the o-minimal singular cohomology. This is done in Section 4. In Section 5 we deal with the model theoretic issue of the invariance of (H_c^*, d_c^*) in o-minimal expansions and in elementary extensions of $\mathcal{N}$. The relationship between the o-minimal Euler characteristic E and the Euler-Poincaré characteristic χ_c associated to (H_c^*, d_c^*) is considered in Section 6.

The paper begins with the basic properties of definably locally compact definable sets and, in Section 3, we introduce the o-minimal cohomology theory $(\check{H}^*, \check{d}^*)$ in the category of pairs of definably locally compact definable sets which, as we shall see, is isomorphic to the o-minimal singular cohomology. The classical analogue of this cohomology, that is the Čech cohomology of locally compact subsets of Euclidean neighbourhood retracts, can be non isomorphic to the topological singular cohomology.

We assume that the reader is familiar with the basic notions and facts of o-minimal structures and of o-minimal singular homology and cohomology. For a treatment of this, see [vdd], [Wo] and [ew] respectively.

2 Definably locally closed sets

In this section we present the basic properties of definably locally compact definable sets that will be useful later. This is the definable analogue of the theory of [d] Chapter IV, Section 8.

Proposition 2.1 *Let $Z \subseteq N^n$ be a definable subset. Then the following are equivalent:*

(1) Z is of the form $C \cap U$ where C is a closed definable subset of N^n and U is an open definable subset of N^n.

(2) There is a definable retraction $r : V \longrightarrow Z$ where V is an open definable subset of N^n.

(3) There is a definable family $\{U_z : z \in Z\}$ of definable open subsets of N^n such that for every $z \in Z$ we have $z \in U_z$ and $U_z \cap Z$ is closed in U_z.

(4) There is a definable family $\{V_z : z \in Z\}$ of definable subsets of N^n such that for every $z \in Z$ the set V_z is a definably compact definable neighbourhood of z in Z.

Proof. If (1) holds then Z is closed in U and (2) follows from [vdd] Chapter VIII, Proposition 3.3. If (2) holds, then regarding r as a definable map $r : V \longrightarrow V$, we have $Z = \overline{Z} \cap V$ since $Z = \overline{Z} = \{v \in V : r(v) = v\}$. Thus (1) holds.

Assume (4). Then $V_z = Z \cap W_z$ where $\{W_z : z \in Z\}$ is a definable family of open definable subsets of N^n. For $z \in Z$, let $U_z = \mathring{W}_z$. Then $Z \cap U_z = V_z \cap U_z$ is closed in U_z. Therefore, the definable family $\{U_z : z \in Z\}$ satisfies (3).

If (3) holds, then we have $Z \cap U_z = \overline{Z} \cap U_z$, hence $Z = Z \cap (\cup\{U_z : z \in Z\}) = \cup\{\overline{Z} \cap U_z : z \in Z\} = \overline{Z} \cap (\cup\{U_z : z \in Z\})$. This implies (1) since $U = \cup\{U_z : z \in Z\}$ is an open definable subset of N^n.

Finally we show that (1) implies (4). Let $\{B_z : z \in Z\}$ be a family of definably compact definable neighbouhoods of $z \in Z$ in N^n. For $z \in Z$, let $W_z = Z \cap B_z = C \cap B_z$. Then each W_z is definably compact and the definable family $\{W_z : z \in Z\}$ shows (4). □

A definable set $Z \subseteq N^n$ satisfying one the conditions of Proposition 2.1 will be called *definably locally closed* or *definably locally compact*.

The next remark is standard. For the semi-algebraic analogue, see [BCR] Proposition 2.2.9.

Remark 2.2 Every definably locally closed definable set $Z \subseteq N^n$ is definably homeomorphic to a closed definable subset of N^{n+1}.

In fact, let C and U be closed and open definable subsets of N^n such that $Z = C \cap U$. Then the definable map $h : U \longrightarrow N^n \times N$ given by $h(x) = (x, \frac{1}{d(x,N^n-U)})$ where $d(x, N^n - U) = \inf\{|x - y| : y \in N^n - U\}$, is a definable embedding of U into N^{n+1} (the projection $\pi : N^{n+1} \longrightarrow N$ onto the first n-coordinates is the inverse of h). Moreover, $h(U) = \{(x, y) \in N^n \times N : y \cdot d(x, N^n - U) = 1\}$ is a closed definable subset of N^{n+1}. Hence, $h(Z)$ is closed in $h(U)$ and hence in N^{n+1}.

As in [BCR] Proposition 2.5.9, one can also show that every definably locally closed definable set $Z \subseteq N^n$ has an o-minimal Alexandrov compactification.

The following proposition is a generalization of [vdd] Chapter VIII, Corollary 3.10 and it was proved in the semi-algebraic case in [dk2]. See also [BCR] Proposition 2.6.9.

Proposition 2.3 *Let Z be a definable subset of N^n, Y a definable set, B a definable subset of Y and $f : B \longrightarrow Z$ a continuous definable map. If Z and B are definably locally closed, then f extends to a continuous definable map from an open definable neighbourhood of B in Y. In particular, if $f, g : B \longrightarrow Z$ are definably homotopic definable continuous maps, then they extend to definably homotopic definable continuous maps from an open definable neighbourhood of B in Y.*

Proof. By replacing Y with an open definable neighbourhood of B in Y if necessary, we can assume that B is closed in Y. Proposition 2.1, there is a definable retraction $r : V \longrightarrow Z$ where V is an open definable subset of N^n. Let $i : Z \longrightarrow V$ be the inclusion. Then by [vdd] Chapter VIII, Corollary 3.10, there is a definable continuous map $f' : Y \longrightarrow N^n$ extending $i \circ f : B \longrightarrow N^n$. Put $W = (f')^{-1}(V)$ and let $\widehat{f} : V \longrightarrow Z$ be given by $\widehat{f} = r \circ f'$. Clearly, $\widehat{f}$ is a definable continuous map extending f to V and $B \subseteq V$.

Now suppose that $F : B \times [0,1] \longrightarrow Z$ is a definable homotopy between f and g. By the first part of the proposition, let W be an open definable neighbourhood of B in Y and let $f', g' : W \longrightarrow Z$ be definable continuous maps extending f and g respectively. Consider the closed definable subset $(W \times 0) \cup (B \times [0,1]) \cup (W \times 1)$ of $W \times [0,1]$ and the definable continuous map G on this set into N^n given by $G(y,0) = i \circ f'(y)$, $G(y,1) = i \circ g'(y)$ for $y \in V$ and $G(y,t) = i \circ F(y,t)$ for $(y,t) \in B \times [0,1]$. By [vdd] Chapter VIII, Corollary 3.10, there is a continuous definable map $\widehat{G} : W \times [0,1] \longrightarrow N^n$ extending G. Let $U = \{y \in W : \widehat{G}(y,[0,1]) \subseteq V\}$. Then U is an open definable neighbourhood of B in Y and $\widehat{F} : U \times [0,1] \longrightarrow Z$ given by $\widehat{F}(y,t) = r \circ \widehat{G}(y,t)$ is a definable continuous map extending F. Moreover, $\widehat{f} = \widehat{F}_{|U \times 0}$ and $\widehat{g} = \widehat{F}_{|U \times 1}$ are extension of f and g to U respectively and $\widehat{F}$ is a definable homotopy between $\widehat{f}$ and $\widehat{g}$. □

Proposition 2.4 *Let Z be a definable subset of N^n which is definably locally closed. If $f, g : Y \longrightarrow Z$ are definable continuous maps and B is a definable subset of Y such that $f_{|B} = g_{|B}$, then there is an open definable neighbourhood U of B in Y and a definable homotopy $F : U \times [0,1] \longrightarrow X$ between $f_{|U}$ and $g_{|U}$ such that for all $x \in B$ and $t \in [0,1]$ we have $F(x,t) = f(x)$.*

Proof. By Proposition 2.1, there is a definable retraction $r : V \longrightarrow Z$ where V is an open definable subset of N^n. Let $i : Z \longrightarrow V$ be the inclusion and let $U = \{y \in Y : \{(1-t)i \circ f(y) + ti \circ g(y) : t \in [0,1]\} \subseteq V\}$. Then U is an open definable neighbourhood of B in Y. Define $F : U \times [0,1] \longrightarrow X$ by $F(y,t) = r((1-t)i \circ f(y) + ti \circ g(y))$. □

Corollary 2.5 *Let $B \subseteq Z$ be a definable subsets of N^n which are definably locally closed. Then there is a definable neighbourhood retract $r : V \longrightarrow B$ in Z and there is an open definable neighbourhood W of B in V such that $i \circ r_{|W}$*

is definably homotopic to the inclusion $j : W \longrightarrow V$*, where* $i : B \longrightarrow V$ *is the inclusion.*

Proof. By Proposition 2.1 one can assume that V is open in N^n. Hence V is definably locally closed. Now apply Proposition 2.4 with $f = i \circ r$ and $g = 1_V$. □

3 The o-minimal cohomology theory $(\breve{H}^*, \breve{d}^*)$

Below we denote by C the category whose objects are pairs (K, L) with $L \subseteq K$ definably locally closed sets and $f : (K, L) \longrightarrow (M, N)$ is a morphism of C if and only if $f : K \longrightarrow M$ is a continuous definable map such that $f(L) \subseteq N$.

For $(K, L) \in \mathrm{Obj}C$, let $\Lambda(K, L)$ be the set of all pairs (U, V) of open definable neighbourhoods of (K, L) directed with reverse inclusion. Set $\Lambda(K) = \Lambda(K, \emptyset)$.

The next construction follows its classical analogue in [d] Chapter VIII, Section 6.

Theorem 3.1 *There is a cohomology theory* $(\breve{H}^*, \breve{d}^*)$ *on* C *with coefficients in* R *such that, for* $(K, L) \in \mathrm{Obj}C$*, the cohomology* R*-module* $\breve{H}^*(K, L; R)$ *is the direct limit* $\lim_{(U,V)\in\Lambda(K,L)} H^q(U, V; R)$*.*

Proof. We start by constructing the homomorphism $\breve{H}^*(f) = \breve{f} : \breve{H}^*(K, L; R) \longrightarrow \breve{H}^*(K', L'; R)$ associated to $f : (K', L') \longrightarrow (K, L) \in \mathrm{Mor}C$. Let $(V, W) \in \Lambda(K, L)$ and, by Proposition 2.3, let $F : U' \longrightarrow K$ be a definable continuous map extending f. Then $(F^{-1}(V), F^{-1}(W)) \in \Lambda(K', L')$ and we have the commutative diagram

$$\begin{array}{ccc} H^q(V, W; R) & \overset{(F_{V,W})^*}{\rightarrow} & H^q(F^{-1}(V), F^{-1}(W); R) \\ \downarrow & & \downarrow \\ H^q(V_1, W_1; R) & \overset{(F_{V_1,W_1})^*}{\rightarrow} & H^q(F^{-1}(V_1), F^{-1}(W_1); R) \end{array}$$

where the vertical arrows are induced by inclusions $(V_1, W_1) \longrightarrow (V, W)$ of elements in $\Lambda(K, L)$ and $F_{V,W} = F_{|(F^{-1}(V), F^{-1}(W))}$.

This diagram goes to the limit and we obtain $\check{f} : \check{H}^*(K,L;R) \longrightarrow \check{H}^*(K',L';R)$. It remains to show that this homomorphism is independent of the choice of F. So let $G : V' \longrightarrow X$ be another definable continuous map extending f. We may assume that $V' = U'$. Moreover, by Proposition 2.4, we may also assume that F and G are definably homotopic. But then, by the homotopy axiom for o-minimal cohomology, we have $(F_{V,W})^* = (G_{V,W})^*$. Therefore, $\check{f}$ is independent of the choice of F.

We now construct the connecting homomorphism $\check{d}^* : \check{H}^q(L;R) \longrightarrow \check{H}^{q+1}(K,L;R)$. Let $(V,W) \in \Lambda(K,L)$. Then, by naturality of d^*, we have the following commutative diagram

$$\begin{array}{ccc} H^q(W;R) & \stackrel{d^*}{\rightarrow} & H^{q+1}(V,W;R) \\ \downarrow & & \downarrow \\ H^q(W_1;R) & \stackrel{d^*}{\rightarrow} & H^{q+1}(V_1,W_1;R) \end{array}$$

where the vertical arrows are induced by inclusions $(V_1,W_1) \longrightarrow (V,W)$ of elements in $\Lambda(K,L)$. This diagram goes to the limit and we obtain the connecting homomorphism $\check{d}^* : \check{H}^q(L;R) \longrightarrow \check{H}^{q+1}(K,L;R)$ which by [d] Chapter VIII, Proposition 5.12 is a natural transformation.

It remains to prove the Eilenberg-Steenrod axiom for cohomology in the category C.

Proof of the exactness axiom. If $(V,W) \in \Lambda(K,L)$, then by the exactness axiom for the o-minimal singular cohomology and naturality of d^* we get the following commutative diagram

$$\begin{array}{ccccccccc} \rightarrow H^q(V;R) & \rightarrow & H^q(W;R) & \stackrel{d^*}{\rightarrow} & H^{q+1}(V,W;R) & \rightarrow & H^{q+1}(V;R) \rightarrow \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \rightarrow H^q(V_1;R) & \rightarrow & H^q(W_1;R) & \stackrel{d^*}{\rightarrow} & H^{q+1}(V_1,W_1;R) & \rightarrow & H^{q+1}(V_1;R) \rightarrow \end{array}$$

where the vertical arrows are induced by inclusions $(V_1,W_1) \longrightarrow (V,W)$ of elements in $\Lambda(K,L)$. This diagram goes to the limit to give, by [d] Chapter VII, Corollary 5.21, the exactness axiom for $(\check{H}^*, \check{d}^*)$.

Proof of the homotopy axiom. If $f,g : (K',L') \longrightarrow (K,L) \in \mathrm{Mor}C$ are definably homotopic, then the argument at the beginning of the proof shows that $\check{f} = \check{g}$.

Proof of the excision axiom. Suppose that U is an open definable subset of K such that $\overline{U} \subseteq \mathring{L}$. Let $X_1 = L$ and $X_2 = K - U$. Consider the obvious morphisms from $\Lambda(X_1) \times \Lambda(X_2)$ into $\Lambda(X_1)$, $\Lambda(X_2)$, $\Lambda(X_1 \cup X_2)$ and $\Lambda(X_1 \cap X_2)$. These morphisms are surjective and hence cofinal. For the first three this is obvious. We now verify the surjectivity of the last morphism. The conditions on K, L and U imply that $X_1 \cap X_2$ separates X_1, X_2 i.e., $\overline{X_2 - X_1} \cap (X_1 - X_2) = \emptyset = \overline{X_1 - X_2} \cap (X_2 - X_1)$. Given $W \in \Lambda(X_1 \cap X_2)$, let $V_1 = W \cup U_1$ and $V_2 = W \cup U_2$ where $U_1 = \{x \in X_1 : \min\{|x-z| : z \in X_1 - X_2\} < \min\{|x-z| : z \in X_2 - X_1\}\}$ and $U_2 = \{x \in X_2 : \min\{|x-z| : z \in X_2 - X_1\} < \min\{|x-z| : z \in X_1 - X_2\}\}$. Then $(V_1, V_2) \in \Lambda(X_1) \times \Lambda(X_2)$ and its image is W.

Now given $(V_1, V_2) \in \Lambda(X_1) \times \Lambda(X_2)$ we have by the excision axiom for the o-minimal singular cohomology $H^*(V_1 \cup V_2, V_1; R) \simeq H^*(V_2, V_1 \cap V_2; R)$. This goes to the limit to give, by [d] Chapter VIII, Corollary 5.21, the isomorphism $\check{H}^*(K, L; R) \longrightarrow \check{H}^*(K - U, L - U; R)$ since $X_1 \cup X_2 = K$ and $X_1 = L$, $X_2 = K - U$ and $X_1 \cap X_2 = L - U$.

Proof of the dimension axiom. Suppose that K is a one point set. And let $(V, W) \in \Lambda(K, \emptyset)$. Then there is a pair $(U, \emptyset) \in \Lambda(K, \emptyset)$ such that $U \subseteq V$ and U is an open ball in N^n. Hence we have $H^q(V, W; R) \longrightarrow H^q(U, \emptyset; R) = H^q(K, \emptyset; R)$. So $\check{H}^q(K, \emptyset; R) = H^q(K, \emptyset; R)$ and $\check{H}^q(K, \emptyset; R)$ is R for $q = 0$ and zero otherwise as required. □

For $(K, \emptyset) \in \mathrm{Obj}C$, we will use $\check{H}^q(K; R)$ to denote $\check{H}^q(K, \emptyset; R)$. The classical analogue of our next theorem is treated in [d] Chapter VIII, Proposition 6.12.

Theorem 3.2 *The cohomology theories* $(\check{H}^*, \check{d}^*)$ *and* (H^*, d^*) *on* C *are isomorphic.*

Proof. Given $(K, L) \in \mathrm{Obj}C$, passing the inclusion homomorphisms $H^q(V, U; R) \longrightarrow H^q(K, L; R)$, with $q \in \mathbb{Z}$ and $(U, V) \in \Lambda(K, L)$, to the limit we get a sequence $(\kappa^q)_{q \in \mathbb{Z}}$ of natural transformations $\kappa^q : \check{H}^q \longrightarrow H^q$ making the diagram

$$\begin{array}{ccc} \check{H}^q(L; R) & \xrightarrow{\check{d}^q} & \check{H}^{q+1}(K, L; R) \\ \downarrow^{\kappa^q} & & \downarrow^{\kappa^{q+1}} \\ H^q(L; R) & \xrightarrow{d^q} & H^{q+1}(K, L; R) \end{array}$$

commutative for all $q \in \mathbb{Z}$.

Let $(A, \emptyset) \in \mathrm{Obj}C$ and take $q \in \mathbb{Z}$. We will show that $\kappa^q : \check{H}^q(A; R) \longrightarrow H^q(A; R)$ is an isomorphism. By Proposition 2.1 there is a definable retraction $r : V \longrightarrow A$ of an open definable neighbourhood V of A. Let $i : A \longrightarrow V$ be the inclusion. Then $H^q(ri) =$ identity, and so $H^q(i)$ is an epimorphism, hence κ^q is also an epimorphism.

By Corollary 2.5, there is an open definable neighbourhood W of A contained in V such that if $j : W \longrightarrow V$ is the inclusion, then there is a definable homotopy between $i \circ r_{|W}$ and j. Hence, $H^q(j) = H^q(r_{|W})H^q(i)$ and we have a factorisation

$$\begin{array}{ccccc} H^q(U'; R) & \rightarrow & H^q(A; R) & & \\ \downarrow & & \uparrow & & \\ H^q(V; R) & \longrightarrow & H^q(A; R) & \longrightarrow & H^q(W; R) \end{array}$$

where U' is any open definable neighbourhood of A containing V. It follows that any class in $H^q(U'; R)$ going to zero in $H^q(A; R)$ goes to zero in $H^q(W; R)$ and thus κ^q is a monomorphism.

The fact that $\kappa^q : \check{H}^q \longrightarrow H^q$ is an isomorphism of cohomologies follows now from the exactness axiom for $(\check{H}^*, \check{d}^*)$ and (H^*, d^*). $\square$

We have the following continuity for the o-minimal cohomology $(\check{H}^*, \check{d}^*)$ which can be proved by Theorem 3.2 or as in [d] Chapter VIII, 6.18 using the purely algebraic criteria in [d] Chapter VIII, 5.18.

Remark 3.3 Let $(K, L) \in$ ObjC and suppose that Λ is a subset of ObjC such that:

(i) Λ is directed under reverse inclusion;

(ii) if $(M, N) \in \Lambda$, then $K \subseteq M$ and $L \subseteq N$;

(iii) if (V, W) is a pair of open definable susbets such that $K \subseteq V$ and $L \subseteq W$, then there is $(M, N) \in \Lambda$ such that $M \subseteq V$ and $N \subseteq W$.

Then $(\check{H}^*(M, N))_{(M,N)\in\Lambda}$ together with the inclusion homomorphisms is a directed system with homomorphisms $\sigma_{M,N} : \check{H}^*(M, N) \longrightarrow \check{H}^*(K, L)$ induced by the inclusion in (ii). The direct limit of these homomorphisms $\sigma : \lim \check{H}^*(M, N) \longrightarrow \check{H}^*(K, L)$ is an isomorphism.

4 The o-minimal cohomology theory (H_c^*, d_c^*)

The classical analogue of the next theorem is treated in [d] pages 288-291 with very few details.

For $(K, L) \in \mathrm{Obj}C$, let $\Omega_c(K, L)$ be the set of all $U \in \mathrm{Obj}C$ such that U is open in K, $L \subseteq U \subseteq K$ and $K - U$ is definably compact. Then $\Omega_c(K, L)$ is direct under reversed inclusion. Let C_c be the subcategory of C such that $\mathrm{Obj}C_c = \mathrm{Obj}C$ and $f : (K, L) \longrightarrow (M, N) \in \mathrm{Mor}C_c$ if and only if for all $U \in \Omega_c(M, N)$ we have $f^{-1}(U) \in \Omega_c(K, L)$.

Theorem 4.1 *There is a cohomology theory (H_c^*, d_c^*) on C_c with coefficients in R such that, for $(K, L) \in \mathrm{ObjC}_c$, the R-module $H_c^*(K, L; R)$ is the direct limit $\lim_{U \in \Omega_c(K,L)} H^q(K, U; R)$ and for $f : (K, L) \longrightarrow (M, N) \in \mathrm{Mor}C_c$ the homomorphism $f_c^* : H_c^*(M, N; R) \longrightarrow H_c^*(K, L; R)$ is the limit homomorphism of the family $(f^*)_U : H^*(M, U; R) \longrightarrow H^*(K, f^{-1}(U); R)$ with $U \in \Omega_c(M, N)$.*

Proof. The functor H_c^* is well defined since by [d] Chapter VIII, 5.9, the family $(f^*)_U : H^*(M, U; R) \longrightarrow H^*(K, f^{-1}(U); R)$ with $U \in \Omega_c(M, N)$ passes to the limit.

It remains to prove the Eilenberg-Steenrod axiom for cohomology in the category C.

Proof of the exactness axiom. If $U \in \Omega_c(K, L)$, $V \in \Omega_c(K, \emptyset)$ and $V \subseteq U$, then by the exactness axiom for triples for the o-minimal singular cohomology and naturality of d^* we get the following commutative diagram

$$\begin{array}{ccccccccc} \rightarrow H^q(K, V; R) & \rightarrow & H^q(U, V; R) & \xrightarrow{d^*} & H^{q+1}(K, U; R) & \rightarrow & H^{q+1}(K, V; R) \rightarrow \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \rightarrow H^q(K, V_1; R) & \rightarrow & H^q(U_1, V_1; R) & \xrightarrow{d^*} & H^{q+1}(K, U_1; R) & \rightarrow & H^{q+1}(K, V_1; R) \rightarrow \end{array}$$

where $U_1 \in \Omega_c(K, L)$, $V_1 \in \Omega_c(K, \emptyset)$, $V_1 \subseteq U_1$, $U_1 \subseteq U$ and $V_1 \subseteq V$ and, the vertical arrows are induced by inclusions. Since, by Remark 3.3, $\lim H^*(U, V) = \lim H^*(L, V)$, this diagram goes to the limit to give, by [d] Chapter VII, Corollary 5.21, the exactness axiom for (H_c^*, d_c^*).

Proof of the homotopy axiom. If $f, g : (K', L') \longrightarrow (K, L) \in \mathrm{Mor}C_c$ are definably homotopic, $(f^*)_U = (g^*)_U : H^*(K', U; R) \longrightarrow H^*(K, f^{-1}(U); R)$ for every $U \in \Omega_c(K', L')$. Thus $f_c^* = g_c^*$.

Proof of the excision axiom. Let $(K,L) \in \mathrm{Obj}C_c$ and suppose that U is an open definable subset of K such that $\overline{U} \subseteq \mathring{L}$. Consider the morphism $V \mapsto V \cap (K-U) : \Omega_c(K,L) \longrightarrow \Omega_c(K-U, L-U)$. This is well defined since $(K-U)-(V\cap(K-U)) = (K-V)\cap(K-U)$, $K-V$ is definably compact and $K-U$ is closed. This morphism is surjective and hence cofinal since, if $W \in \Omega_c(K-U, L-U)$ and $V = W \cup U$, then $L \subseteq V \subseteq K$, V is open in K (as $K-U$ is closed in K) and $K-V = K-(W\cup U) = (K-U)-W$ is definably compact.

Now as $V-U = V\cap(K-U)$ and $\overline{U} \subseteq \mathring{L} \subseteq L \subseteq V$, by the excision axiom for the o-minimal singular cohomology, we have isomorphisms $H^*(K,V;R) \longrightarrow H^*(K-U, V-U;R)$ induced by the inclusion. Hence, this goes to the limit to give, by [d] Chapter VIII, Corollary 5.21, the isomorphisms $H_c^*(K,L;R) \longrightarrow H_c^*(K-U, L-U;R)$ as required.

Proof of the dimension axiom. Suppose that K is a one point set and let $U \in \Omega_c(K,\emptyset)$. Then $U \in \{\emptyset, K\}$ and so $H_c^*(K) = H^*(K,\emptyset)$. □

We call the cohomology theory of Theorem 4.1 the *o-minimal cohomology with definably compact supports.*

Remark 4.2 For $(K,L) \in$ ObjC, let $\Omega(K,L)$ be the set of all definably locally closed definable subsets A of K such that $L \subseteq A \subseteq K$ and $\overline{K-A}$ is definably compact. Then $\Omega(K,L)$ is direct under reversed inclusion and $H_c^*(K,L;R)$ is the direct limit $\lim_{A\in\Omega(K,L)} H^*(K,A;R)$.

This is proved as in [d] Chapter VIII, 6.23 using the purely algebraic criteria [d] Chapter VIII, 5.18 and Theorem 3.2.

If $\overline{\Omega}(K,L)$ is the set of all closed definable subsets A of K such that $L \subseteq A \subseteq K$ and $\overline{K-A}$ is definably compact, then $\overline{\Omega}(K,L)$ is direct under reversed inclusion and $H_c^*(K,L;R)$ is the direct limit $\lim_{A\in\overline{\Omega}(K,L)} H^*(K,A;R)$. In fact, the morphism $A \mapsto \overline{A} : \Omega(K,L) \longrightarrow \overline{\Omega}(K,L)$ is cofinal (even surjective).

The following observation is also quite useful.

Remark 4.3 If $(K,L) \in$ ObjC and L is closed in K, then by the excision axiom for the o-minimal singular cohomology, we have an isomorphism $H^*(K,U;R) \longrightarrow H^*(K-L, U-L;R)$ for every $U \in \Omega_c(K,L)$. Going to the limit we get, by [d] Chapter VIII, Corollary 5.21, the isomorphisms $H_c^*(K,L;R) \longrightarrow H_c^*(K-L;R)$.

5 Comparation theorems for (H_c^*, d_c^*)

We start we the following result from [ps] which will play an important role.

Remark 5.1 A definable set is definably compact if and only if it is closed and bounded. Hence the notion of definably compact is invariant in o-minimal expansions and in elementary extension. Furthermore, if $\mathcal{N}$ is an o-minimal expansion of the field of real numbers, then a definable set is definably compact if and only if it is compact.

A model $\mathcal{M}$ of the theory $\mathrm{Th}_N(\mathcal{N})$ determines the categories $C(\mathcal{M})$ and $C_c(\mathcal{M})$ and determines an obvious functor

$$\mathcal{M} : C \longrightarrow C(\mathcal{M})$$

which sends (X, A) into $(X(M), A(M))$ and sends $f : (X, A) \longrightarrow (Y, B)$ into $f^M : (X(M), A(M)) \longrightarrow (Y(M), B(M))$.

For a model $\mathcal{M}$ of $\mathrm{Th}_N(\mathcal{N})$ let $\Omega_c(\mathcal{M})(X, A)$ be the set of all $U \in \mathrm{Obj}C(\mathcal{M})$ such that U is open in X, $A \subseteq U \subseteq X$ and $X - U$ is definably compact in $\mathcal{M}$. By Remark 5.1 restriction gives a morphism $\mathcal{M} : \Omega_c(X, A) \longrightarrow \Omega_c(\mathcal{M})(X(M), A(M))$. This is not cofinal: take for example, X the closed unit ball centered at the origin, A the origin and U an infinitesimal open ball in $\mathcal{M}$ centered at the origin.

Claim 5.2 *Suppose that $\mathcal{M}$ is a model of $\mathrm{Th}(\mathcal{N})$. Then the restriction also gives a functor $\mathcal{M} : C_c \longrightarrow C_c(\mathcal{M})$. Furthermore, for every $q \in \mathbb{Z}$ and $(X, A) \in \mathrm{Obj}C_c$, the system $(H^q(X(M), U(M); R))_{U \in \Omega_c(X,A)}$ is cofinal in $(H^q(X(M), U; R))_{U \in \Omega_c(\mathcal{M})(X(M),A(M))}$.*

Proof. Consider $f : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}C_c$ and suppose that $f^M : (X(M), A(M)) \longrightarrow (Y(M), B(M))$ is not in $\mathrm{Mor}C_c(\mathcal{M})$. Then there is an open $\mathcal{M}$-definable subset U of $Y(M)$ such that $B(M) \subseteq U \subseteq Y(M)$, $Y(M) - U$ is $\mathcal{M}$-definably compact and $X(M) - (f^M)^{-1}(U)$ is not $\mathcal{M}$-definably compact. By Remark 5.1, this last condition is first-order. Hence, there is an open definable subset V of Y defined over N such that $B \subseteq V \subseteq Y$, $Y - V$ is definably compact and $X - f^{-1}(V)$ is not definably compact. But this contradicts the fact that $f : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}C_c$. Therefore, the restriction gives a functor $\mathcal{M} : C_c \longrightarrow C_c(\mathcal{M})$.

We now show that the system $(H^q(X(M),U(M);R))_{U\in\Omega_c(X,A)}$ is cofinal in $(H^q(X(M),U;R))_{U\in\Omega_c(\mathcal{M})(X(M),A(M))}$. For this we will prove that given $U\in\Omega_c(\mathcal{M})(X(M),A(M))$ there is $V\in\Omega_c(X,A)$ such that $H^q(X(M),U;R)$ is isomorphic to $H^q(X(M),V(M);R)$.

So let $U\in\Omega_c(\mathcal{M})(X(M),A(M))$. Then by Remark 5.1, there are definable sets $S\subseteq N^{p+q}$ and $T\subseteq N^q$ such that projection map $\pi:N^{p+q}\longrightarrow N^q$ sends S onto T, $U=((\pi_{|S})^M)^{-1}(m)$ for some $m\in T(M)$ and, for every $t\in T(M)$, the $\mathcal{M}$-definable set $((\pi_{|S})^M)^{-1}(t)$ over t is an open $\mathcal{M}$-definable subset of $X(M)$ such that $A(M)\subseteq((\pi_{|S})^M)^{-1}(t)\subseteq X(M)$ and $X(M)-((\pi_{|S})^M)^{-1}(t)$ is $\mathcal{M}$-definably compact.

Applying the definable trivialization theorem ([vdd] Chapter IX, Theorem 1.7) to $\pi_{|S}:S\longrightarrow T$, we see that there is a definable set F and a definable map $\lambda:S\longrightarrow F$ such that $(\pi_{|S},\lambda):S\longrightarrow T\times F$ is a definable homeomorphism. In particular, for every $t\in T$, $(\pi_{|S},\lambda)_{|}:(\pi_{|S})^{-1}(t)\longrightarrow\{t\}\times F$ is a definable homeomorphism. Consequently, if $n\in T$ and $V=(\pi_{|S})^{-1}(n)$, then $V\in\Omega_c(X,A)$ (by Remark 5.1) and $V(M)$ and U are $\mathcal{M}$-definably homeomorphic to $F(M)$. This implies that $H^*(V(M);R)\simeq H^*(U;R)$. By the exactness axiom for o-minimal singular cohomology and the five lemma ([d] Chapter I, (2.9)), it follows that $H^*(X(M),V(M);R)\simeq H^*(X(M),U;R)$ as required. □

Theorem 5.3 *Suppose that* $\mathcal{M}$ *is a model of* Th($\mathcal{N}$), $(X,A)\in \mathrm{Obj}C_c$ *and* $f:(X,A)\longrightarrow(Y,B)\in\mathrm{Mor}C_c$. *Then we have isomorphisms*

$$i^*_{c\mathcal{M}}:H^*_c(X,A;R)\longrightarrow H^*_c(X(M),A(M);R)$$

making the following diagram

$$\begin{array}{ccc} H^*_c(Y,B;R) & \xrightarrow{f^*_c} & H^*_c(X,A;R) \\ \downarrow i^*_{c\mathcal{M}} & & \downarrow i^*_{c\mathcal{M}} \\ H^*_c(Y(M),B(M);R) & \xrightarrow{(f^M)^*_c} & H^*_c(X(M),A(M);R) \end{array}$$

commutative.

Proof. This follows imediately from Claim 5.2 together with the invariance of o-minimal singular cohomology in elementary extensions proved in [ew]. □

Let $\mathcal{M}$ be an o-minimal expansion of $\mathcal{N}$ and denote by $C^{(\mathcal{M})}$ and $C_c^{(\mathcal{M})}$ the categories C and C_c computed in $\mathcal{M}$. The model $\mathcal{M}$ determines an obvious functor

$$(\mathcal{M}) : C \longrightarrow C^{(\mathcal{M})}$$

which sends (X, A) into $(X^{(M)}, A^{(M)})$ and sends $f : (X, A) \longrightarrow (Y, B)$ into $f^{(M)} : (X^{(M)}, A^{(M)}) \longrightarrow (Y^{(M)}, B^{(M)})$.

For $\mathcal{M}$ an o-minimal expansion of $\mathcal{N}$, let $\Omega_c^{(\mathcal{M})}(X, A)$ be the set of all $U \in \mathrm{Obj}C^{(\mathcal{M})}$ such that U is open in X, $A \subseteq U \subseteq X$ and $X - U$ is definably compact in $\mathcal{M}$. Restriction gives a morphism $(\mathcal{M}) : \Omega_c(X, A) \longrightarrow \Omega_c^{(\mathcal{M})}(X^{(M)}, A^{(M)})$.

Claim 5.4 *Suppose that $\mathcal{M}$ is an o-minimal expansion of $\mathcal{N}$ and $(X, A) \in \mathrm{Obj}C_c$. Then the morphism $(\mathcal{M}) : \Omega_c(X, A) \longrightarrow \Omega_c^{(\mathcal{M})}(X^{(M)}, A^{(M)})$ is cofinal, restriction gives a functor $(\mathcal{M}) : C_c \longrightarrow C_c^{(\mathcal{M})}$ and, for every $q \in \mathbb{Z}$, the directed system $(H^q(X^{(M)}, U^{(M)}; R))_{U \in \Omega_c(X,A)}$ is cofinal in the directed system $(H^q(X^{(M)}, U; R))_{U \in \Omega_c^{(\mathcal{M})}(X^{(M)}, A^{(M)})}$.*

Proof. By Remark 2.2, we may assume without loss of generality that X is a closed definable subset of N^n.

Let $U \in \Omega_c^{(\mathcal{M})}(X^{(M)}, A^{(M)})$. Then U is an open $\mathcal{M}$-definable subset of $X^{(M)}$, $A^{(M)} \subseteq U \subseteq X^{(M)}$ and $X^{(M)} - U$ is $\mathcal{M}$-definably compact. Consider the distance function $d(-, A^{(M)}) : X^{(M)} - U \longrightarrow M$. Since $X^{(M)} - U$ is $\mathcal{M}$-definably compact there is $\epsilon > 0$ such that $d(x, A^{(M)}) \geq \epsilon$ for all $x \in X^{(M)} - U$.

For $0 < \delta < \epsilon$, let $V_\delta = \{x \in X : d(x, A) < \delta\}$. Then V_δ is an open definable subset of X, $A \subseteq V_\delta \subseteq X$ and $V_\delta^{(M)} \subseteq U$.

Since $X^{(M)} - U$ is $\mathcal{M}$-definably compact, by Remark 5.1, $X^{(M)} - U$ is closed and bounded. Hence there is a closed (and bounded) definable box B in N^n such that $X^{(M)} - U \subseteq B^{(M)}$. Since X is closed in N^n, by Remark 5.1, the set $B^{(M)} \cap X^{(M)}$ is $\mathcal{M}$-definably compact. Let $V = V_\delta \cup (X - X \cap B)$. Then V is an open definable subset of X, $A \subseteq V \subseteq X$ and $V^{(M)} \subseteq U$. Moreover, $X^{(M)} - V^{(M)} = (X^{(M)} - V_\delta^{(M)}) \cap (B^{(M)} \cap X^{(M)})$ is $\mathcal{M}$-definably compact. By Remark 5.1 again, we conclude that $X - V$ is definably compact. Consequently, there is $V \in \Omega_c(X, A)$ such that $V^{(M)} \in \Omega_c^{(\mathcal{M})}(X^{(M)}, A^{(M)})$ and $V^{(M)} \subseteq U$ as required.

We now verify that restriction gives a functor $(\mathcal{M}) : C_c \longrightarrow C_c^{(\mathcal{M})}$. In fact, consider $f : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}C_c$ and suppose that $f^{(M)} :$

$(X^{(M)}, A^{(M)}) \longrightarrow (Y^{(M)}, B^{(M)})$ is not in Mor$C_c^{(\mathcal{M})}$. Then there is $U \in \Omega_c^{(\mathcal{M})}(Y^{(M)}, B^{(M)})$ such that $X^{(M)} - (f^{(M)})^{-1}(U)$ is not $\mathcal{M}$-definably compact. But then, there is $V \in \Omega_c(Y, B)$ such that $V^{(M)} \in \Omega_c^{(\mathcal{M})}(Y^{(M)}, B^{(M)})$ and $V^{(M)} \subseteq U$. Since, $X^{(M)} - (f^{(M)})^{-1}(U) \subseteq X^{(M)} - (f^{(M)})^{-1}(V^{(M)})$, it follows that $X^{(M)} - (f^{(M)})^{-1}(V^{(M)})$ is not $\mathcal{M}$-definably compact. By Remark 5.1, this implies that $X - f^{-1}(V)$ is not definably compact. But this contradicts the fact that $f : (X, A) \longrightarrow (Y, B) \in$ MorC_c.

Since the morphism $(\mathcal{M}) : \Omega_c(X, A) \longrightarrow \Omega_c^{(\mathcal{M})}(X^{(M)}, A^{(M)})$ is cofinal, for every $q \in \mathbb{Z}$, the system $(H^q(X^{(M)}, U^{(M)}; R))_{U \in \Omega_c(X,A)}$ is cofinal in the directed system $(H^q(X^{(M)}, U; R))_{U \in \Omega_c^{(\mathcal{M})}(X^{(M)}, A^{(M)})}$. □

Using Claim 5.4 instead of Claim 5.2, we can prove invariance results in this context analogous to the invariance results in elementary extensions proved above.

Suppose that $\mathcal{N}$ is an o-minimal expansion of the field of real numbers and denote by $C^{(top)}$ denote the category of locally closed pairs of topological spaces with continuous maps. Then we have an obvious functor

$$(top) : C \longrightarrow C^{(top)}$$

sending maps (X, A) into $(X(\mathbb{R}), A(\mathbb{R}))$ and $f : (X, A) \longrightarrow (Y, B)$ into $f^{\mathbb{R}} : (X(\mathbb{R}), A(\mathbb{R})) \longrightarrow (Y(\mathbb{R}), B(\mathbb{R}))$.

For $\mathcal{N}$ an o-minimal expansion of the field of real numbers, let $\Omega_c^{(top)}(X, A)$ be the set of all $U \in$ Obj$C^{(top)}$ such that U is open in X, $A \subseteq U \subseteq X$ and $X - U$ is compact. By Remark 5.1, restriction gives a morphism $(top) : \Omega_c(X, A) \longrightarrow \Omega_c^{(top)}(X(\mathbb{R}), A(\mathbb{R}))$.

Claim 5.5 *Suppose that $\mathcal{N}$ is an o-minimal expansion of the field of real numbers and $(X, A) \in$ ObjC_c. Then the morphism $(top) : \Omega_c(X, A) \longrightarrow \Omega_c^{(top)}(X(\mathbb{R}), A(\mathbb{R}))$ is cofinal, restriction gives a functor $(top) : C_c \longrightarrow C_c^{(top)}$ and, for every $q \in \mathbb{Z}$, the system $(H^q(X(\mathbb{R}), U(\mathbb{R}); R))_{U \in \Omega_c(X,A)}$ is cofinal in $(H^q(X(\mathbb{R}), U; R))_{U \in \Omega_c^{(top)}(X(\mathbb{R}), A(\mathbb{R}))}$.*

Claim 5.5 can be proved by replacing in the proof of Claim 5.4 the category $C^{(\mathcal{M})}$ by the category $C^{(top)}$. Using Claim 5.5 instead of Claim 5.2, we can prove invariance results in this context analogous to the invariance results in elementary extensions proved above.

6 O-minimal Euler characteristics

The *Grothendieck semi-ring* $SK_0(C)$ of the category C of definably locally closed definable sets is the quotient of the free abelian semigroup over the symbols $[X]$ with $X \in \mathrm{Obj}C$ by the relations

(1) $[\emptyset] = 0$,

(2) $[X] = [Y]$ if there is a definable homeomorphism between X and Y and

(3) $[X] = [X \setminus Y] + [Y]$ if Y is a closed definable subset of X.

The product of objects of C induces the ring structure

(4) $[X] \cdot [Y] = [X \times Y]$.

The *Grothendieck ring* $K_0(C)$ of the category C is the unique minimal ring that embeds the quotient semi-ring obtained from $SK_0(C)$ by identifying $[X]$ and $[Y]$ if and only if there is $[Z] \in SK_0(C)$ such that $[X] + [Z] = [Y] + [Z]$.

Definition 6.1 A *generalized Euler characteristic* on C, with values in a ring S, is map $e : \mathrm{Obj}C \longrightarrow S$ such that $e = e' \circ [\ ]$, where $[\ ] : \mathrm{Obj}C \longrightarrow SK_0(C)$ is the obvious map, and $e' : SK_0(C) \longrightarrow S$ is a simi-ring homomorphism. In particular, the following hold:

(1) $e(\emptyset) = 0$,

(2) $e(X) = e(Y)$ if there is a definable homeomorphism between X and Y,

(3) $e(X) = e(X \setminus Y) + e(Y)$ if Y is a closed definable subset of X and

(4) $e(X \cdot Y) = e(X \times Y)$.

The map $[\ ] : \mathrm{Obj}C \longrightarrow K_0(C)$ is a generalized Euler characteristic on C called the *universal generalized Euler characteristic* on C.

For example, the o-minimal Euler characteristic $E : \mathrm{Obj}C \longrightarrow \mathbb{Z}$ as defined in [vdd] is a generalized Euler characteristic on C. We now use the o-minimal cohomology theory (H_c^*, d_c^*) to construct another generalized Euler characteristic on C.

Definition 6.2 Suppose that $X \in \mathrm{Obj}C$. The *o-minimal Euler-Poincaré characteristic with definably compact supports* $\chi_c(X)$ *of* X is by definition $\chi_c(X) = \sum_{i=0}^{\dim X} (-1)^i b_i$ where $b_i = \dim_{\mathbb{Q}} H_c^i(X; \mathbb{Q})$ are the *Betti numbers of* X.

Theorem 6.3 *The o-minimal Euler-Poincaré characteristic with definably compact supports χ_c is the universal generalized Euler characteristic on C and for every definably locally compact definable set X we have $E(X) = \chi_c(X)$.*

Proof. Let $\Phi : X \longrightarrow |K|$ be a definable triangulation of X such that the vertices of K are defined over $\mathbb{Q}$. Then $\chi_c(X) = \chi_c(|K|)$ and, by the invariance results in Section 5, $\chi_c(|K|) = \chi_c^{(top)}(|K|(\mathbb{R}))$ where $\chi_c^{(top)}$ is the Euler-Poincaré characteristic of the locally compact set $|K|(\mathbb{R})$ with respect to the topological cohomology with compact supports and coefficient in $\mathbb{Q}$. On the other hand, we have $E(X) = E(|K|) = E(|K|(\mathbb{R})) = \sum_{\sigma \in K}(-1)^{\dim \sigma}$. Thus it remains to show that $\chi_c^{(top)}(|K|(\mathbb{R})) = E(|K|(\mathbb{R}))$.

First note that by the exactness axiom and the Künneth formula for the topological cohomology with compact supports and coefficient in $\mathbb{Q}$, see [d] Chapter VIII, 8.19, $\chi_c^{(top)}$ is a generalized Euler characteristic on $C^{(top)}$.

Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic subset. Then by [BCR] 2.3.6, S is semi-algebraically homeomorphic to a finite disjoint union $B_{d_1} \cup \cdots \cup B_{d_m}$ where for each $i = 1, \ldots, m$, $B_{d_i} = (-1, 1)^{d_i}$. Since $\chi_c^{(top)}(B_{d_i}) = (-1)^{d_i} = E(B_{d_i})$, it follows that $\chi_c^{(top)}(S) = E(S)$. In particular, $\chi_c^{(top)}(|K|(\mathbb{R})) = E(|K|(\mathbb{R}))$. □

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