Analytic surface germs with minimal Pythagoras number

José F. Fernando*

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Abstract

We determine all complete intersection surface germs whose Pythagoras number is 2, and find they are all embedded in $\mathbb{R}^3$ and have the property that every positive semidefinite analytic function germ is a sum of squares of analytic function germs. In addition, we discuss completely these properties for *mixed* surface germs in $\mathbb{R}^3$. Finally, we find in higher embedding dimension three different families with these same properties.

1 Introduction

In the investigation of *sums of squares* in analytic surface germs, the property that every *positive semidefinite* function germ is a sum of squares (in short $\mathcal{P} = \Sigma$) has appeared closely connected to the minimal value of the *Pythagoras number* p. Here we will refer always to the *analytic* Pythagoras number, that is, the smallest integer $p \geq 1$ such that every sum of squares of *analytic* function germs is a sum of p squares of *analytic* function germs. This invariant is always finite for surface germs ([Fe1]) and infinite for germs of higher dimension ([Fe3]).

Back to our properties $\mathcal{P} = \Sigma$ and $p = 2$, they were first compared in [Rz2], where a small list of candidates for them was produced. Later, in [Fe2], we saw that in fact that list gives all *unmixed* surface germs in $\mathbb{R}^3$ with $\mathcal{P} = \Sigma$, and all have $p = 2$. In this paper we single out the invariant p, and look for surface germs with $p = 2$. Our main result is proved in Section 2:

Theorem 1.1 *The complete intersection germs of dimension* ≥ 2 *with* $p[X] = 2$ *are exactly the following*

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(i) $z^2 - x^3 - y^5 = 0$ *(Brieskorn's singularity).*

(ii) $z^2 - x^3 - xy^3 = 0.$

(iii) $z^2 - x^3 - y^4 = 0.$

(iv) $z^2 - x^2 = 0$ *(two transversal planes).*

(v) $z^2 - x^2 - y^2 = 0$ *(cone).*

(vi) $z^2 - x^2 - y^k = 0,\ k \geq 3$ *(deformations of two planes).*

(vii) $z^2 - x^2 y = 0$ *(Whitney's umbrella)*

(viii) $z^2 - x^2 y + y^3 = 0.$

(ix) $z^2 - x^2 y - (-1)^k y^k = 0,\ k \geq 4$ *(deformations of Whitney's umbrella).*

Since we already know that all the germs of the list above have $p = 2$, the esential goal here is the converse, that is, *every complete intersection with* $p = 2$ *belongs to the list.* Note that complete intersection curve germs have all $p \leq 2$ because they are planar curve germs.

This theorem together with [Fe2] shows that the properties $\mathcal{P} = \Sigma$ and $p = 2$ are equivalent for *unmixed* surface germs in $\mathbb{R}^3$. Mixed surface germs in $\mathbb{R}^3$ are unions of surface germs with some irreducible components of dimension 1; these are exactly the surface germs in $\mathbb{R}^3$ which are not complete intersections, and very few of them have the properties under consideration. Namely, we prove

Theorem 1.2 *The mixed surface germs in* $\mathbb{R}^3$ *with* $p = 2$ *are either*

(a) *the union of a plane and a transversal line, and then* $\mathcal{P} = \Sigma$*, or*

(b) *the union of a plane and a transversal singular planar curve, and then* $\mathcal{P} \neq \Sigma$*.*

This requires an extremely careful analysis that we present in Section 3. Thus, we close completely the case of surface germs in $\mathbb{R}^3$. We want to stress here that although very predictable, as we naively conjectured at the *2001 Rennes International Congress of Real Analytic and Algebraic Geometry 2001*, the actual proofs are far from easy. The difficulty is due to the fact that sos's which are not sums of two squares are very rare (not generic) in the mixed surface cases that one is lead to analize.

In higher embedding dimension there are many more possibilities. First of all, it is easy to produce *reducible* surface germs with $\mathcal{P} = \Sigma$ and $p = 2$: If $X, Y \subset \mathbb{R}^3$ have the properties, then $Z = (X \times \{0\}) \cup (\{0\} \times Y) \subset \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ has them too. Note that, every non-unit $f \in \mathbb{R}\{x_1, x_2, x_3, y_1, y_2, y_3\} = \mathbb{R}\{x, y\}$ can be written over Z as $f(x, 0) + f(0, y)$ (all $x_i y_j$'s vanish on Z), so that $f \in \mathcal{P}(Z)$ if and only if $f(x, 0) \in \mathcal{P}(X), f(0, y) \in \mathcal{P}(Y)$.

Therefore, if $f \in \mathcal{P}(Z)$, $f \equiv f(x,0) + f(0,y) \equiv a_1^2 + a_2^2 + b_1^2 + b_2^2 \equiv (a_1+b_1)^2 + (a_2+b_2)^2$ and, we conclude $\mathcal{P} = \Sigma$ and $p = 2$ for Z.

Hence, we concentrate on *irreducible* germs and produce in Section 4:

Example 1.3 The *Veronese cones* $X_n \subset \mathbb{R}^{n+1}, n \geq 2$ (cones over the rational normal curve), which are the surface germs given by the equations

$$F_{ij} = x_i x_j - x_{i-1}x_{j+1} = 0, \quad 1 \leq i \leq j \leq n-1,$$

and whose complexifications are parametrized by $\gamma(z,w) = (z^n, z^{n-1}w, \dots, zw^{n-1}, w^n)$, (see [Ha]). It is easy to prove that X_n has multiplicity n and embedding dimension $n+1$. For these surface germs $\mathcal{P} = \Sigma$ and $p = 2$, which we shortly denote by $\mathcal{P} = \Sigma_2$ (Th. 4.1). These X_n's are not complete intersections, but they are at least normal, hence Cohen-Macaulay (but not Gorenstein). In particular, $X_2 \subset \mathbb{R}^3$ is the usual cone $x_1^2 = x_0x_2$, already settled in [FeRz],[Fe2].

Example 1.4 The family of *generalized Whitney's umbrellas* $Y_n \subset \mathbb{R}^{n+1}$, $n \geq 2$, which are the analytic closures of the set germs parametrized by

$$\varphi_n : (s,t) \mapsto (s, st, \dots, st^{n-1}, t^n) = (x_0, x_1, \dots, x_{n-1}, x_n).$$

It is not difficult to check that the ideal of Y_n is generated by the polynomials

$$x_i x_j - x_0 x_\ell x_n^q : \quad i + j = qn + \ell \quad \text{and} \quad \begin{array}{l} 1 \leq i \leq j \leq n-1 \\ 0 \leq \ell \leq n-1, \end{array}$$

and Y_n consists of the union of the image of φ_n and the x_n-axis; again mult $= n$ and emb dim $= n+1$. These surface germs have also $\mathcal{P} = \Sigma_2$ (Th. 4.4). However, these Y_n's are not complete intersections. In fact, they are neither normal (x_1/x_0 is integral over $\mathcal{O}(Y_n)$) nor Gorenstein (by Stanley's Criterion, [Ei, 21.14], [St]). On the positive, they are Cohen-Macaulay: $\operatorname{depth}(Y_n) \leq \dim(Y_n) = 2$ and $\{x_0, x_n\}$ is a regular sequence. The first umbrella $Y_2 \subset \mathbb{R}^3$ is the classical Whitney umbrella $x_1^2 = x_0^2 x_2$, for which we already knew $\mathcal{P} = \Sigma_2$ ([Rz2]).

Example 1.5 A family of irreducible surface germs $Z_n \subset \mathbb{R}^{n+1}$, $n \geq 3$, parametrized by

$$\phi_n : (s,t) \mapsto (x_0, \dots, x_n) = (s, st, \dots, st^{n-2}, t^{n-1}, t^n),$$

with $p = 2$ and $\mathcal{P} \neq \Sigma$ (Th.4.5), mult $= n$ and emb dim $= n+1$. The surface germ Z_n is given by the equations

$$\left\{ \begin{array}{ll} x_k^{n-1} - x_0^{n-1} x_{n-1}^k & k = 1, \dots, n-2 \\ x_0 x_n^k - x_k x_{n-1}^k & k = 1, \dots, n-2 \\ x_n^{n-1} - x_{n-1}^n. & \end{array} \right.$$

These Z_n's cannot be complete intersections by Th 1.1, but in fact, it is not difficult to verify that they are not even Cohen-Macaulay and, therefore, not Gorenstein (the general hyperplane section given by $x_0 - x_{n+1} = 0$ contains an embedded point given by the ideal $(x_0, x_{n-1}, x_i x_j, x_i x_n, x_n^{n-1} : 1 \le i \le j \le n-2)$ and, hence, $\operatorname{depth} Z_n < \dim Z_n$). Notice that for $n = 2$, $Z_2 \subset \mathbb{R}^3$ would be the regular germ $x_2 = x_1^2$ for which of course $\mathcal{P} = \Sigma_2$ ([BR]).

We finish here with several questions that arise naturally from the above results and examples:

Open questions. *(1)* Is there in higher embedding dimension any analytic germ with $\mathcal{P} = \Sigma$ and $p \neq 2$?

(2) Are there *very singular* surfaces (worse than our normal X_n's and our Cohen-Macaulay Y_n's) with $\mathcal{P} = \Sigma$ and $p = 2$?

(3) Are there *very regular* surfaces (better than our not Cohen-Macaulay Z_n's) with $p = 2$ and $\mathcal{P} \neq \Sigma$?

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2 Proof of the main result

The purpose of this section is to prove Theorem 1.1. Let X be an analytic set germ (at the origin of $\mathbb{R}^n$); we denote by $\mathcal{O}(X)$ the ring of germs of analytic functions on X. If $X \subset \mathbb{R}^n$ we have $\mathcal{O}(X) = \mathbb{R}\{x_1, \dots, x_n\}/\mathcal{J}(X)$, where $\mathcal{J}(X)$ is the ideal of all analytic function germs vanishing on X. We recall that a germ $f \in \mathcal{O}(X)$ is *positive semidefinite* or *psd* if it is ≥ 0 on X; we denote by $\mathcal{P}(X)$ the set of all psd's of X. We will denote by $\Sigma(X)$ the set of all sums of squares of elements of $\mathcal{O}(X)$. Morever, $p[X]$ stands for the Pythagoras number of $\mathcal{O}(X)$.

Lemma 2.1 Let $a_1, a_2, b_1, b_2 \in \mathbb{R}\{x, y\}$. If $\omega(b_1) = 0$, there exist $\alpha_1, \alpha_2, \beta_1 \in \mathbb{R}\{x, y\}$ such that

$$(a_1 + zb_1)^2 + (a_2 + zb_2)^2 = (\alpha_1 + z\beta_1)^2 + \alpha_2^2.$$

Proof. Just take $\alpha_1 = \dfrac{a_1 b_1 + a_2 b_2}{\sqrt{b_1^2 + b_2^2}}$, $\alpha_2 = \dfrac{a_1 b_2 - a_2 b_1}{\sqrt{b_1^2 + b_2^2}}$ and $\beta_1 = \sqrt{b_1^2 + b_2^2}$. ■

After this we turn to our main result:

Proof of Theorem 1.1. We begin by remarking that in [Rz2, 1.1,2.1], the author actually proves that if $X \subset \mathbb{R}^n$ is a complete intersection of dimension ≥ 2 and $p[X] = 2$ then X is analytically equivalent to a surface germ in $\mathbb{R}^3$ of equation $z^2 = F(x, y)$. In what follows, we will see in several steps that a surface germ $X: \; z^2 - F(x, y) = 0$ with $p = 2$ is, in fact, one of the surfaces of the list.

(A) First, we prove that $\omega(F) \le 3$. If $\omega(F) \ge 4$ then $F = Q + G$ where $\omega(G) \ge 5$ and Q is either 0 or a homogeneous polynomial of degree 4. Let

$$G = (x^2 + az)^2 + (y^2 + bz)^2 + (xy + cz)^2.$$

We claim there are $a, b, c \in \mathbb{R}$ not all 0 such that G is not a sum of two squares in $\mathcal{O}(X)$. Otherwise, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{R}\{x, y\}$ with $\omega(\beta_2) \ge 1$ (maybe $\beta_2 = 0$, see 2.1) such that

$$G_p \equiv x^4 + y^4 + x^2y^2 + 2(ax^2 + by^2 + cxy)z + (a^2 + b^2 + c^2)F$$
$$= (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 - \gamma(z^2 - F)$$

and therefore, comparing coefficients with respect to z:

0) $x^4 + y^4 + x^2y^2 + (a^2 + b^2 + c^2)F = \alpha_1^2 + \alpha_2^2 + \gamma F$

1) $ax^2 + by^2 + cxy = \alpha_1\beta_1 + \alpha_2\beta_2$

2) $0 = \beta_1^2 + \beta_2^2 - \gamma$

From *0)* we deduce that $\omega(\alpha_1), \omega(\alpha_2) \ge 2$ and from *1)* that $\omega(\beta_1) = 0$, and therefore $\beta_1(0) = \lambda \ne 0$. Thus, if we compare initial forms in *0)* and *1)* we deduce that

0) $x^4 + y^4 + x^2y^2 + (a^2 + b^2 + c^2)Q = \mathrm{In}(\alpha_1)^2 + \mathrm{In}(\alpha_2)^2 + \lambda^2 Q$

1) $ax^2 + by^2 + cxy = \mathrm{In}(\alpha_1)\lambda$

and hence

$$\lambda^2(x^4 + y^4 + x^2y^2 + (a^2 + b^2 + c^2 - \lambda^2)Q)$$
$$= (ax^2 + by^2 + cxy)^2 + (ux^2 + vy^2 + wxy)^2 \quad (*)$$

where $\lambda, u, v, w \in \mathbb{R}$. We have to distinguish three cases:

(A1) $Q = 0$. If we take $a = 0, b = 1, c = 1$ we have

$$\lambda^2(x^4 + y^4 + x^2y^2) = (y^2 + xy)^2 + (ux^2 + vy^2 + wxy)^2.$$

and hence, $u^2 = \lambda^2 \ne 0, uw = 0, 2 + 2vw = 0$ which is impossible. Thus, this G is not a sum of two squares. ■

If $Q \neq 0$, after a linear change, we can suppose $Q = \varepsilon x^4 + q_3 x^2 y^2 + q_4 xy^3 + q_5 y^4$ where $\varepsilon = \pm 1$ and $q_3, q_4, q_5 \in \mathbb{R}$.

(A2) If $q_4 = 0$ another linear change allows us to suppose $Q = \varepsilon x^4 + q_3 x^2 y^2 + q_5 y^4$ with $(q_3, q_5) \neq (\pm 1, 0), (\pm 0, 1)$. Consider the non zero polynomial in b, c:

$$P(b,c) = (\varepsilon + b^2 + c^2)(\varepsilon(\varepsilon - q_3)(\varepsilon - q_5) - b^2 q_5(\varepsilon - q_3) - c^2 q_3(\varepsilon - q_5)).$$

Take $a = 0$ and $b, c \neq 0$ such that $P(b,c) \neq 0$. Comparing coefficients in $(*)$ we obtain

$x^4)$ $\lambda^2(1 + (b^2 + c^2 - \lambda^2)\varepsilon) = u^2$

$y^4)$ $\lambda^2(1 + (b^2 + c^2 - \lambda^2)q_5) = b^2 + v^2$

$x^2y^2)$ $\lambda^2(1 + (b^2 + c^2 - \lambda^2)q_3) = c^2 + 2uv + w^2$

$x^3y)$ $0 = uw$

$xy^3)$ $0 = bc + vw.$

Since $bc \neq 0$ then $v, w \neq 0$ and $u = 0$. Therefore we deduce that $\lambda^2 = \varepsilon + b^2 + c^2$ and $w = -bc/v$. Plugging these values in the equations above we have

$$\begin{aligned} (\varepsilon + b^2 + c^2)(1 - \varepsilon q_5) - b^2 &= v^2 \\ ((\varepsilon + b^2 + c^2)(1 - \varepsilon q_3) - c^2)v^2 &= b^2 c^2 \end{aligned}$$

And then, combining the two equations, we conclude that $P(b,c)$ must be zero, against our choice. ■

(A3) If $q_4 \neq 0$ another linear change allows us to suppose $Q = \varepsilon x^4 + q_3 x^2 y^2 + 2xy^3 + q_5 y^4$. We take $c = 0, a = 0, b \neq 0$ if $\varepsilon(q_3 + q_5) - q_3 q_5 \geq 0$ and $c = 0, a = 2b \neq 0$ if $\varepsilon(q_3 + q_5) < q_3 q_5$. Comparing coefficients in $(*)$ we get

$x^4)$ $\lambda^2(1 + (a^2 + b^2 - \lambda^2)\varepsilon) = a^2 + u^2$

$y^4)$ $\lambda^2(1 + (a^2 + b^2 - \lambda^2)q_5) = b^2 + v^2$

$x^2y^2)$ $\lambda^2(1 + (a^2 + b^2 - \lambda^2)q_3) = 2ab + 2uv + w^2$

$x^3y)$ $0 = uw$

$xy^3)$ $\lambda^2(a^2 + b^2 - \lambda^2) = vw.$

If $w = 0$ then $\lambda^2 = a^2 + b^2$ and we have $u^2 = b^2, v^2 = a^2, a^2 + b^2 = 2ab + 2uv$. Therefore $a^2 + b^2 = 2ab \pm 2ab$ which is impossible by our choice $a = 0 \neq b$ or $a = 2b \neq 0$. Thus,

$w \neq 0$ and $u = 0$. Substituting $u = 0$ and $\lambda^2(a^2 + b^2 - \lambda^2) = vw$ in the equations above we have

$$\begin{aligned} \lambda^2 + \varepsilon vw &= a^2 \\ \lambda^2 + q_5 vw &= b^2 + v^2 \\ \lambda^2 + q_3 vw &= 2ab + w^2. \end{aligned}$$

Now substituting $\lambda^2 = a^2 - \varepsilon vw$ we obtain

$$\begin{aligned} a^2 - b^2 &= v(v - (q_5 - \varepsilon)w) \\ 0 = a(a - 2b) &= w(w - (q_3 - \varepsilon)v). \end{aligned}$$

Since $w \neq 0$ then $w = (q_3 - \varepsilon)v, v \neq 0$ and thus $a^2 - b^2 = v^2(\varepsilon(q_3 + q_5) - q_3 q_5)$. But this is impossible with our choice of a, b. ■

Thus, $\omega(F) \leq 3$. Once we have **(A)**, we find some order two and order three restrictions.

(B) If $\omega(F) = 2$ then, after a linear change, we can suppose that the equation of X is of the kind $z^2 + \varepsilon x^2 - y^k$ with $\varepsilon = \pm 1$, $k \geq 2$. If $k = 2$, then $z^2 + \varepsilon x^2 - y^2 = 0$ is the equation of the cone, which belongs to the list. Here we show that ε must be -1 for $k \geq 3$, hence X belongs to the list.

If $\varepsilon = 1$ the function germ $(z + x^2)^2 + x^2 + y^2$ is not a sum of two squares in $\mathcal{O}(X)$. Indeed, were it so, there would exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{R}\{x, y\}$ such that $\omega(\beta_2) \geq 1$ (maybe $\beta_2 = 0$, see 2.1) and

$$(z + x^2)^2 + x^2 + y^2 \equiv 2zx^2 + x^4 + y^2 + y^k = (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 - \gamma(z^2 + x^2 - y^k)$$

Comparing coefficients with respect to z we obtain that

0) $x^4 + y^2 + y^k + \gamma(x^2 - y^k) = \alpha_1^2 + \alpha_2^2$

1) $x^2 = \alpha_1\beta_1 + \alpha_2\beta_2$

2) $0 = \beta_1^2 + \beta_2^2 - \gamma$.

If $\omega(\beta_1) \geq 1$ then, by *2)*, $\omega(\gamma) \geq 1$ and by *0)* $\alpha_i = \lambda_i y + g_i$ where $\lambda_i \in \mathbb{R}$, $\lambda_1^2 + \lambda_2^2 = 1$, $g_i \in (x, y)^2$, which is impossible by *1)*; hence $\omega(\beta_1) = 0$. Combining the equations we have

$$\beta_1^2(x^4 + y^2 + y^k + (\beta_1^2 + \beta_2^2)(x^2 - y^k)) = (x^2 - \alpha_2\beta_2)^2 + \alpha_2^2\beta_1^2,$$

and computing a little we conclude that

$$\beta_1^2((\beta_1^2 + \beta_2^2)x^2 + y^2) + (\beta_1^2 - 1)x^4 + \beta_1^2(1 - \beta_1^2 - \beta_2^2)y^k = \alpha_2((\beta_1^2 + \beta_2^2)\alpha_2 - 2x^2\beta_2),$$

but this is impossible because the power series on the left is irreducible, since its initial form is $a^2x^2+b^2y^2$ with $a,b\neq 0$. ■

(C) If $\omega(F)=3$, then, after a linear change, the initial form of F is xy^2,$x^2y\pm y^3$ or x^3. Again, we study several cases:

(C1) If $\mathrm{In}(F)=x^2y$ or $x^2y\pm y^3$ then, after a change of coordinates (standard classification of singularities), we can suppose that F is one of the following power series: x^2y, $x^2y\pm y^k$, $k\geq 3$. If F is x^2y or $x^2y+(-1)^ky^k$ then X belongs to the list. Now, if $F=x^2y-(-1)^ky^k$ we show that there exist sums of squares of analytic function germs on $X_k : z^2=x^2y-(-1)^ky^k$ which are not sums of 2 squares.

We begin with the surface germ $X_3 : z^2-y(x^2+y^2)$. Let $f=g_6+z^2(x^2+y^2)$ where g_6 is a homogeneous polynomial in x,y of degree 6 which is an sos and $(x^2+y^2)\not| \, g_6$ (e.g. x^6). If f was a sum of two squares, then there would exist $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma\in\mathbb{R}\{x,y\}$ such that

$$f\equiv g_6+(x^2+y^2)^2y=(\alpha_1+z\beta_1)^2+(\alpha_2+z\beta_2)^2-\gamma(z^2-(x^2+y^2)y),$$

and so

0) $g_6+(x^2+y^2)^2y=\alpha_1^2+\alpha_2^2+(\beta_1^2+\beta_2^2)(x^2+y^2)y,$

1) $0=\alpha_1\beta_1+\alpha_2\beta_2.$

Comparing orders in *0)* we deduce that $\omega(\beta_1^2+\beta_2^2-(x^2+y^2))\geq 3$. Thus, $\mathrm{In}(\beta_1^2+\beta_2^2)=x^2+y^2$ and we conclude that $\omega(\beta_i)=1$, so that the series β_1,β_2 are relatively prime. Therefore, by *1)*, there exist a series $d\in\mathbb{R}\{x,y\}$ such that $\alpha_1=\beta_2d,\alpha_2=-\beta_1d$. Plugging these in *0)* we get

$$g_6=d^2(\beta_1^2+\beta_2^2)+y(x^2+y^2)q$$

where $q\in\mathbb{R}\{x,y\}$ is a series of order ≥ 3. Finally, comparing initial forms in this expression, we conclude that $(x^2+y^2)|g_6$ which is impossible by hypothesis.

Next, we prove using suitable blowings-up that $p[X_k]\geq 3$ if $k\geq 4$. Suppose $p[X_{k_0}]=2$ for some $k_0\geq 4$. We consider the algebraic surfaces

$$S_{X_{k_0}} \text{ of equation } v^2-u^2y+(-1)^{k_0}y^{k_0}=0, \quad S_{X_3} \text{ of equation } z^2-x^2y-y^3=0$$

and the biregular equivalence

$$\begin{array}{rrcl} \varphi: & S_{X_{k_0}}\setminus\{y=0\} & \to & S_{X_3}\setminus\{y=0\} \\ & (u,y,v) & \mapsto & (x,y,z)=\begin{cases} \left(\dfrac{u}{y^\ell},y,\dfrac{v}{y^\ell}\right) & \text{if } k_0=2\ell+3 \\ \left(\dfrac{v}{y^{\ell+1}},y,\dfrac{u}{y^\ell}\right) & \text{if } k_0=2\ell+4. \end{cases} \end{array}$$

Now, let $f=\sum_i(c_i+zd_i)^2$ be a sum of squares in X_3 such that $c_i,d_i\in\mathbb{R}[x,y]$ for all i. Consider

$$f\circ\varphi=\frac{g(u,y,v)}{y^{2r}}$$

where $g\in\mathbb{R}[u,y,v]$ is clearly a sum of squares in $\mathcal{O}(X_{k_0})$. Since $p[X_{k_0}]=2$, there exist $\alpha,\beta,\gamma\in\mathbb{R}\{u,y,v\}$ such that $g=\alpha^2+\beta^2+(v^2-u^2y+(-1)^{k_0}y^{k_0})\gamma$. Composition with φ^{-1} gives

$$y^{2r}f=\alpha(xy^\ell,y,zy^\ell)^2+\beta(xy^\ell,y,zy^\ell)^2+y^{2\ell}(z^2-x^2y-y^3)\gamma(xy^\ell,y,zy^\ell)\quad \text{if } k_0=2\ell+3$$

$$y^{2r}f=\alpha(zy^\ell,y,xy^{\ell+1})^2+\beta(zy^\ell,y,xy^{\ell+1})^2 + y^{2\ell}(z^2-x^2y-y^3)\gamma(zy^\ell,y,xy^{\ell+1})\quad \text{if } k_0=2\ell+4$$

Thus, there exist $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1\in\mathbb{R}\{x,y\}$ such that

$$y^{2r}f\equiv y^{2r}\left(\sum_i c_i^2+y(x^2+y^2)\sum_i d_i^2+2z\sum_i c_id_i\right) = (\alpha_1+z\alpha_2)^2+(\beta_1+z\beta_2)^2-\gamma_1(z^2-y(x^2+y^2))\quad (*)$$

and comparing coefficients we get

0) $y^{2r}\left(\sum_i c_i^2+y(x^2+y^2)\sum_i d_i^2\right)=\alpha_1^2+\beta_1^2+y(x^2+y^2)\gamma_1$

1) $\sum_i c_id_i=\alpha_1\alpha_2+\beta_1\beta_2$

2) $\gamma_1=\alpha_2^2+\beta_2^2$.

By *0)* $y|\alpha_1,\beta_1$ and then $y|\gamma_1$; by *2)* $y|\alpha_2,\beta_2$ and thus $y^2|\gamma_1$. Hence, we can divide the expression $(*)$ by y^2 and continue the argument until we end up with and expression of f as a sum of 2 squares in $\mathcal{O}(X_3)$. This shows $p[X_3]=2$ for polynomials, and using the M. Artin's Approximation Theorem ([Ar],[JP]) one deduces by a standard argument ([Fe2]) that $p[X_3]=2$, contradiction. ■

(C2) If $\mathrm{In}(F)=x^3$ and $p[X]=2$ then, after a change of coordinates, F is equal to x^3+xy^3, x^3+y^4 or x^3+y^5, so X belongs to the list.

Indeed, since $\mathrm{In}(F)=x^3$ then (changing x by $-x$ if necessary) there exist a Weierstrass polynomial $P=x^3+p_1(y)y^2x^2+p_2(y)y^3x+p_3(y)$ ($p_i\in\mathbb{R}\{y\}$) and a unit $U\in\mathbb{R}\{x,y\}$ such that $\mathrm{In}(P)=x^3$, $U(0,0)>0$ and $F=PU$; after the change of coordinates $(x,y,z)\mapsto(x-p_1(y)y^2/3,y,\sqrt{U}z)$, we can suppose that the equation of X is of the type $z^2-x^3-a(y)y^3x-b(y)y^4$ for some $a,b\in\mathbb{R}\{y\}$. Let $\varepsilon>0$ be such that the series $\phi=z^2-x^3-a(y)y^3x-b(y)y^4$ converges on the set $\mathbb{R}\times(-\varepsilon,\varepsilon)\times\mathbb{R}$, and let

$$S_{X,\varepsilon}=\{(x,y,z)\in\mathbb{R}\times(-\varepsilon,\varepsilon)\times\mathbb{R}:\phi=0\}.$$

After this preparation we proceed in several steps:

(a) If $\omega(a) \geq 1$ and $\omega(b) \geq 2$ then $p[X] \geq 3$.

Let $Y:\ v^2 - yH = 0$, where $H = u^3 - a(y)yu - b(y)y$ and $S_{Y,\varepsilon}$ as above. Let φ be the biregular map

$$\begin{aligned} \varphi:\ \ S_{X,\varepsilon} \setminus \{y=0\} \ &\rightarrow\ S_{Y,\varepsilon} \setminus \{y=0\} \\ (x,y,z)\ &\mapsto\ (u,y,v) = \left(\frac{x}{y}, y, \frac{z}{y}\right). \end{aligned}$$

In our hypothesis $\omega(H) \geq 3$ and then, as we have seen above, $Y: v^2 - yH = 0$ has Pythagoras number ≥ 3. However, if $p[X] = 2$ then, arguing as in ***(C1)***, we can deduce that $p[Y] = 2$, which is impossible.

Indeed, let $f = \sum_i (c_i + vd_i)^2$ be an sos in Y such that $c_i, d_i \in \mathbb{R}[u,y]$ for all i. Consider the composition

$$f \circ \varphi = \frac{g(x,y,z)}{y^{2r}}$$

where $g \in \mathbb{R}[u,y,v]$ is clearly an sos in $\mathcal{O}(X)$. Since $p[X] = 2$, there exist $\alpha, \beta, \gamma \in \mathbb{R}\{x,y,z\}$ such that $g = \alpha^2 + \beta^2 + (z^2 - x^3 - a(y)y^3x - b(y)y^4)\gamma$. Composing with φ^{-1} we obtain

$$y^{2r} f = \alpha(uy,y,vy)^2 + \beta(uy,y,vy)^2 + y^2(v^2 - yH)\gamma(uy,y,vy)$$

Now, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1 \in \mathbb{R}\{x,y\}$ such that

$$\begin{aligned} y^{2r} f \equiv y^{2r} &\left(\sum_i c_i^2 + yH \sum_i d_i^2 + 2v \sum_i c_i d_i \right) \\ &= (\alpha_1 + v\beta_1)^2 + (\alpha_2 + v\beta_2)^2 - \gamma_1(v^2 - yH) \quad (*) \end{aligned}$$

and comparing coefficients we have

0) $y^{2r}\left(\sum_i c_i^2 + yH \sum_i d_i^2\right) = \alpha_1^2 + \alpha_2^2 + yH\gamma_1$

1) $\sum_i c_i d_i = \alpha_1\alpha_2 + \beta_1\beta_2$

2) $\gamma_1 = \beta_1^2 + \beta_2^2$.

By *0)* $y|\alpha_1, \beta_1$ and then, since y does not divide H, $y|\gamma_1$; by *2)* $y|\alpha_2, \beta_2$ and thus $y^2|\gamma_1$. Hence, we can divide the expression $(*)$ by y^2 and continue the argument until we end up with and expression of f as a sum of 2 squares in $\mathcal{O}(Y)$. Again, by means of M. Artin's Approximation Theorem we pass from polynomials to power series and deduce that $p[Y] = 2$, a contradiction.

Next, we discuss the factorization of $F = x^3 + a(y)y^3x + b(y)y^4$:

(b) If F is the product of three (possibly equal) irreducible factors then $p[X] \geq 3$.

Suppose $F = f_1 f_2 f_3$, where some or all the factors may coincide. Then, since the initial form of F is x^3, we can write $f_k = x + \lambda_k(x,y)$ where $\omega(\lambda_k) \geq 2$ and then

$$\begin{aligned} F &= (x+\lambda_1)(x+\lambda_2)(x+\lambda_3) \\ &= x^3 + x^2(\lambda_1+\lambda_2+\lambda_3) + x(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + (\lambda_1\lambda_2\lambda_3) \\ &= x^3 + a(y)y^3 x + b(y)y^4 \end{aligned}$$

From this equality we deduce

$$b(y)y^4 = F(0,y) = \lambda_1(0,y)\lambda_2(0,y)\lambda_3(0,y), \text{ which has order } \geq 6,$$

$$a(y)y^3 = \frac{\partial F}{\partial x}(0,y) = \sum_{1\leq i<j\leq 3} \lambda_i(0,y)\lambda_j(0,y)\left(1 + \frac{\partial \lambda_k}{\partial x}(0,y)\right), \text{ which has order } \geq 4,$$

and where $1 \leq k \leq 3$, $k \neq i,j$. Hence, $\omega(a) \geq 1, \omega(b) \geq 2$ and, by *(a)*, $p[X] \geq 3$.

(c) If F is reducible and $p[X] = 2$, then $F = x^3 - xy^3$(which is in the list).

By the previous remark, $F = fg$ and f, g must be irreducible, say $\omega(f) = 2, \omega(g) = 1$ and we can suppose $\mathrm{In}(f) = x^2, \mathrm{In}(g) = x$. If f is semidefinite, it is a sum of two squares with initial form x^2 and, after multiplying by a suitable ortogonal matrix, we can suppose $f = (x + \mu_1(x,y))^2 + (\mu_2(x,y))^2$ and $g = x + \mu_3(x,y)$ with $\omega(\mu_k) \geq 2$. Therefore

$$F = (x + \mu_1(x,y) + i\mu_2(x,y))(x + \mu_1(x,y) - i\mu_2(x,y))(x + \mu_3(x,y)),$$

and proceeding similarly to *(b)* (we have again three irreducible factors although two of them are complex) we are in the hypothesis of *(a)* and $p[X] \geq 3$. Hence, if $p[X] = 2$, f should be irreducible and real, so after a change of coordinates, $F = (x^2 - y^k)(x + \mu(x,y))$, $k \geq 3$, $\omega(\mu) \geq 2$. By the Weierstrass Preparation Theorem there exist a series $\alpha \in \mathbb{R}\{y\}$ and a unit $U \in \mathbb{R}\{x,y\}$ such that $x + \mu(x,y) = (x + \alpha(y)y^2)U(x,y)$. Changing x by $-x$ (if necessary) we can suppose $U(0,0) > 0$ and after a change $(x,y,z) \mapsto (x,y,\sqrt{U(x,y)}z)$, the equation of our germ is $z^2 - (x^2 - y^k)(x + \alpha(y)y^2)$. For $k \geq 4$

$$F = x^3 + \alpha(y)x^2y^2 - y^k x - y^{k+2}\alpha(y)$$

and, after the change $x \mapsto x - \alpha(y)y^2/3$, we are again in the conditions of *(a)* so that $p[X] \geq 3$.

Finally, for $k = 3$ we get $F = (x^2 - y^3)(x + \cdots)$ and by classification of singularities, after a change F becomes $x^3 - xy^3$.

(d) If F is irreducible then $p[X] = 2$ if and only if $F = x^3 + y^4$ or $x^3 + y^5$.

Suppose F irreducible. By classification of singularities we can transform F into $x^3 \pm y^4$ or $F = x^3 + xy^4 a'(y) + y^5 b'(y)$. Suppose first $F = x^3 + xy^4 a'(y) + y^5 b'(y)$. If $b'(0) = 0$, by *(a)*, $p[X] \geq 3$. If $b'(0) \neq 0$ then another change makes $F = x^3 + y^5$, and X belongs to the list. After this, the only cases left are $F = x^3 \pm y^4$: if $F = x^3 + y^4$, X is in the list; for $F = x^3 - y^4$ we see that $p[X] \geq 3$.

Indeed, consider the algebraic surfaces $S_X : z^2 - x^3 + y^4 = 0$, $S_Y : v^2 - u^3 y + y^2 = 0$ and the biregular map

$$\begin{array}{rccc} \varphi: & S_X \setminus \{y=0\} & \to & S_Y \setminus \{y=0\} \\ & (x,y,z) & \mapsto & (u,v,w) = \left(\dfrac{x}{y}, y, \dfrac{z}{y}\right). \end{array}$$

As in *(1)* one sees that if $p[X] = 2$ then $p[Y] = 2$. But this is impossible because $Y : v^2 + y^2 - u^3 y = v^2 + (y - u^3/2)^2 - u^6/4 = 0$ is equivalent to $Y' : v^2 + w^2 - u^6 = 0$ which has, as we have seen, Pythagoras number ≥ 3. ■

3 The mixed case

In this section we find all mixed surface germs $X \subset \mathbb{R}^3$ with Pythagoras number 2 and prove that only the simplest one of them has the property $\mathcal{P} = \Sigma$. First of all:

Proposition 3.1 *Let $X \subset \mathbb{R}^3$ be a mixed surface germ. Then $\mathcal{P}(X) = \Sigma(X)$ if and only if X is equivalent to the union of a plane and a transversal line. Furthermore, in this case, $p[X] = 2$.*

Proof. We begin by proving that if X is the union of a plane π and a transversal line ℓ then every $f \in \mathcal{P}(X)$ is a sum of squares of analytic function germs. Indeed, after a change of coordinates the ideal of X is (zx, zy) and, every non unit f in $\mathcal{O}(X)$ can be written uniquely as $f_1(x,y) + f_2(z)$ where $f_1 \in \mathbb{R}\{x,y\}$, $f_2 \in \mathbb{R}\{z\}$; note that $f(0,0) = 0, g(0) = 0$. Now $f = f_1(x,y) + f_2(z) \in \mathcal{P}(X)$ if and only if $f_1 \in \mathcal{P}(\pi)$ and $f_2 \in \mathcal{O}(\ell)$, or equivalently $f_1(x,y) = a(x,y)^2 + b(x,y)^2$ and $f_2(z) = c(z)^2$. Thus, $f = f_1 + f_2 \equiv (a+c)^2 + b^2$ in $\mathcal{O}(X)$, as wanted.

Conversely, if $\mathcal{P}(X) = \Sigma(X)$, by [Fe2, 2.1], $\omega(\mathcal{J}(X)) = 2$. Let I (resp. J) be the ideal of the union of the components of X of dimension 2 (resp. 1). Then $\mathcal{J}(X) = I \cap J$; moreover, since the ideal $I \subset \mathbb{R}\{x,y,z\}$ has height 1, it is principal, and we write $I = (\varphi)$ with $\varphi \in \mathbb{R}\{x,y,z\}$. It is not difficult to check that $\mathcal{J}(X) = I \cdot J$ and therefore $2 = \omega(\mathcal{J}(X)) = \omega(I) + \omega(J)$. Thus, $\omega(I) = \omega(J) = 1$ and, after a change of coordinates, we can suppose that $I = (z)$ and $J = (\psi_1, \psi_2)$ where $\psi_j \in \mathbb{R}\{x,y,z\}$ and $1 = \omega(\psi_1) \leq \omega(\psi_2)$.

We are to prove that after a change of coordinates $J = (x,y)$. To that end, we begin by proving that $\mathcal{Z}(J)$ is regular, hence irreducible. Indeed, suppose $\mathcal{Z}(J)$ singular. Then there exists a function f in $\mathcal{P}(\mathcal{Z}(J)) \setminus \Sigma(\mathcal{Z}(J))$ ([Sch]) and the function $z^2 f \in \mathcal{P}(X)$ is not an sos in $\mathcal{O}(X)$. If it were, there would exist $a_1, \dots, a_p, b_1, b_2 \in \mathbb{R}\{x, y, z\}$ such that

$$z^2 f = a_1^2 + \cdots + a_p^2 + z\psi_1 b_1 + z\psi_2 b_2.$$

From this we see that $z|\, a_i$, say $a_i = z\alpha_i$. Hence, the function $z^2(f - \alpha_1^2 - \cdots - \alpha_p^2)$ vanishes on X, which means that $f - \alpha_1^2 - \cdots - \alpha_p^2 = 0$ on $\mathcal{Z}(J)$ and thus f is an sos, contradiction. Therefore, J is generated by two elements of order 1 with independent initial forms, and after a change of coordinates, we can suppose $I = (z)$, $J = (x, \psi(y, z))$ where $\psi = y$ or $z - y^k$, $k \geq 2\}$.

If $\psi = y$ we are done. So we only have to discard the cases $\psi = z - y^k$, $k \geq 2$. If k is even then $f = z$ is psd on X but it is not an sos. If k is odd we consider the function $f = zy$ which is clearly psd on X but not an sos: if there were $a_1, \dots, a_p, b_1, b_2 \in \mathbb{R}\{x, y, z\}$ such that

$$zy = a_1^2 + \cdots + a_p^2 + zxb_1 + z(z - y^k)b_2,$$

comparing initial forms, there would exist $\lambda, \mu \in \mathbb{R}$ such that the quadratic form $q = zy + \lambda z^2 + \mu zx$ would be a sum of squares of linear forms, but q is not even psd. ■

Now, we characterize the mixed surface germs in $\mathbb{R}^3$ with Pythagoras number 2. First, note that if $Y \subset X$ then $p[Y] \leq p[X]$ (there exist an epimorphism from $\mathcal{O}(Y)$ onto $\mathcal{O}(Y)$). Hence, if Y is a mixed surface germ contained in the union of two transversal planes, which has Pythagoras number 2 ([Rz2], [Fe2]), it also has Pythagoras number 2. The aim of the following theorem is to prove that there are no more mixed surface germs in $\mathbb{R}^3$ with Pythagoras number 2.

Theorem 3.2 *Let $X \subset \mathbb{R}^3$ be a mixed surface germ with Pythagoras number 2. Then X is contained in the union of two transversal planes.*

To prove this, we need the following preliminary lemmas

Lemma 3.3 *Let $X \subset \mathbb{R}^3$ be given by the equations*

$$z(z + 2g) = 0, zf = 0 \text{ (resp. } z^2 - g^2 = 0, (z - g)f = 0)$$

where $f, g \in \mathbb{R}\{x, y\}$ and $\varphi \in \mathcal{O}(X)$: $\varphi \equiv \sum_{i=1}^r (a_i + zb_i)^2$ with $a_i, b_i \in \mathbb{R}\{x, y\}$. Then, there exist $\varphi_1, \varphi_2, q_1, q_2 \in \mathbb{R}\{x, y\}$ such that either

$$\varphi \equiv \varphi_1 + z\varphi_2 = \sum_{i=1}^r (a_i + zb_i)^2 + q_1 z(z + 2g) + q_2 zf$$

$$\text{(resp. } \varphi \equiv \varphi_1 + z\varphi_2 = \sum_{i=1}^r (a_i + zb_i)^2 + q_1(z^2 - g^2) + q_2(z - g)f).$$

Proof. First, suppose $X : z(z+2g) = zf = 0$. Since $z(z+2g) \in \mathbb{R}\{x,y\}[z]$, by Weierstrass division, there exist $\varphi_1, \varphi_2 \in \mathbb{R}\{x,y\}$ such that $\varphi_1 + z\varphi_2 \equiv \varphi \equiv \sum_{i=1}^{r}(a_i + zb_i)^2$. Therefore

$$\varphi_1 + z\varphi_2 = \sum_{i=1}^{r}(a_i + zb_i)^2 + Q_1 z(z+2g) + Q_2 zf,$$

where $Q_1, Q_2 \in \mathbb{R}\{x,y,z\}$. Now, again by Weierstrass division, $Q_2 = Q_3(z+2g) + q_2$ where $q_2 \in \mathbb{R}\{x,y\}$ and $Q_3 \in \mathbb{R}\{x,y,z\}$, and thus:

$$\varphi_1 + z\varphi_2 = \sum_{i=1}^{r}(a_i + zb_i)^2 + (Q_1 + Q_3)z(z+2g) + q_2 zf.$$

On the other hand, in the ring $\mathbb{R}\{x,y\}[z]$ we have $\sum_{i=1}^{r}(a_i + zb_i)^2 = -q_1 z(z+2g) + R$ where $\deg_z R \leq 1, \deg_z q_1 = 0$. By the uniqueness of the Weierstrass division $Q_1 + Q_3 = q_1 \in \mathbb{R}\{x,y\}$ as we wanted.

The case $X : z^2 - g^2 = (z-g)f = 0$ follows from the previous one by the change of coordinates $z \mapsto z + g$. ■

Lemma 3.4 *Let $\varphi \in \mathbb{R}\{t\}$ be a unit, $n \geq 1$. Consider the equation*

$$\lambda(x)^n = \varphi(x\lambda(x))$$

in $\mathbb{R}\{x\}$. If $\varphi(0)$ has an n-root in $\mathbb{R}$, then the previous equation has a solution in $\mathbb{R}\{x\}$.

Proof. Let $b \in \mathbb{R}$ be such that $b^n = \varphi(0)$. The equation above is equivalent to the following one:

$$F(x,y) = (b+y)^n - (\varphi(x(b+y))) = 0$$

which satisties:

$$\begin{aligned} F(0,0) &= 0 \\ \frac{\partial F}{\partial y}(0,0) &= nb^{n-1} \neq 0. \end{aligned}$$

By the Implicit Function Theorem ([JP, 3.3]) there exists a series $\psi \in \mathbb{R}\{x\}$ such that $\psi(0) = 0$ and $F(x, \psi(x)) = 0$. Then take $\lambda = b + \psi$. ■

Proof of Theorem 3.2. Let I (resp. J) be the ideal of the union of components of dimension 2 (resp. 1) of X; $\mathcal{J}(X) = I \cap J$. Using $f = x^2 + y^2 + z^2$ it is not difficult to prove that, since $p[X] = 2$, $\omega(\mathcal{J}(X)) = 2$. As in 3.1, after a change of coordinates, we have $I = (z)$ and $J = (\psi_1, \psi_2)$ where $\psi_j \in \mathbb{R}\{x,y,z\}$ and $1 = \omega(\psi_1) \leq \omega(\psi_2)$.

Now, if the initial form of a ψ_i of order 1 and z are linearly independent, after a change of coordinates, $J = (x, \psi(y, z))$ and we are done. So we can suppose $I = (z), J = (z - 2g(x, y), f(x, y))$ where $f, g \in \mathbb{R}\{x, y\}$ and $\omega(f), \omega(g) \geq 2$. We will find a function germ in $\Sigma_3(X) \setminus \Sigma_2(X)$. We distinguish several cases:

(A) $\omega(f) \geq 3$. Replacing z by $z + g$, we can suppose that the equations of X are: $z^2 - g^2 = 0$, $(z + g)f = 0$. Let

$$G = (Q_1 + az)^2 + (Q_2 + bz)^2 + (Q_3 + cz)^2.$$

where $(a, b, c) \in \mathbb{R}^3 \setminus \{0\}$ and Q_1, Q_2, Q_3 are quadratic forms. Suppose that G is a sum of two squares: by 3.3 there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}\{x, y\}$ with $\omega(\beta_2) \geq 1$ (maybe $\beta_2 = 0$, see 2.1) such that

$$\begin{aligned} G \equiv Q_1^2 + Q_2^2 + Q_3^2 + 2(aQ_1 + bQ_2 + cQ_3)z + (a^2 + b^2 + c^2)g^2 \\ = (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 - \gamma_1(z^2 - g^2) - 2\gamma_2(z + g)f \end{aligned}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}\{x, y\}$ by 3.3. Comparing coefficients with respect to z:

0) $Q_1^2 + Q_2^2 + Q_3^2 + (a^2 + b^2 + c^2)g^2 = \alpha_1^2 + \alpha_2^2 + \gamma_1 g^2 - 2\gamma_2 g f$

1) $aQ_1 + bQ_2 + cQ_3 = \alpha_1\beta_1 + \alpha_2\beta_2 - \gamma_2 f$

2) $0 = \beta_1^2 + \beta_2^2 - \gamma_1$

From *0)* we see that $\omega(\alpha_1), \omega(\alpha_2) \geq 2$ and from *1)* we deduce that $\omega(\beta_1) = 0$, that is, $\beta_1(0) = \lambda \neq 0$. By *2)* $\gamma_1(0) = \lambda^2$ and comparing initial forms in *0)* and *1)* we deduce that

0) $Q_1^2 + Q_2^2 + Q_3^2 + (a^2 + b^2 + c^2)g_2^2 = \mathrm{In}(\alpha_1)^2 + \mathrm{In}(\alpha_2)^2 + \lambda^2 g_2^2$

1) $aQ_1 + bQ_2 + cQ_3 = \mathrm{In}(\alpha_1)\lambda$

where g_2 is the homogeneous component of g of degree 2. Hence, if we take $Q_1 = x^2, Q_2 = y^2, Q_3 = xy$ we obtain

$$\lambda^2 \left(x^4 + y^4 + x^2y^2 + (a^2 + b^2 + c^2 - \lambda^2)g_2^2\right) = (ax^2 + by^2 + cxy)^2 + (ux^2 + vy^2 + wxy)^2$$

for suitable $u, v, w \in \mathbb{R}$. Thus, we obtain the same equations $(*)$ used in part **(A)** of the proof of Th.1.1 and, as it was seen there, they are not always solvable. ■

(B) $\omega(f) = 2$, $\omega(g) = 2$ and their initial forms are linearly independent. We distinguish two cases

(B1) $\mathrm{rk}(\mathrm{In}(f)) = 2$. Then after a change of coordinates we can suppose $f = xy$ (it is not definite because J is the ideal of a curve germ) and $\mathrm{In}(g) = ax^2 + bxy + cy^2$ with $a^2 + c^2 \neq 0$. We have

$$(z(z - 2g), zf) = (z(z - 2g) + 2bzf, zf) = (z(z - 2g'), zf)$$

where $g' = g - bf$ has initial form $\mathrm{In}(g') = ax^2 + cy^2$, and after a linear change in the variables x, y, we can suppose $\mathrm{In}(g) = x^2+y^2, x^2-y^2$ or y^2. Now, let again $G = (Q_1+az)^2+(Q_2+bz)^2+(Q_3+cz)^2 \in \Sigma(X)$ where $(a,b,c) \in \mathbb{R}^3 \setminus \{0\}$ and the Q_i's are quadratic forms. Suppose that G is a sum of two squares: by 3.3 there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}\{x,y\}$ with $\omega(\beta_2) \geq 1$ (maybe $\beta_2 = 0$, see 2.1) such that

$$\begin{aligned} G \equiv Q_1^2 + Q_2^2 + Q_3^2 + 2(aQ_1 + bQ_2 + cQ_3 + (a^2+b^2+c^2)g)z \\ = (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 - \gamma_1(z^2 - 2zg) - 2\gamma_2 zf \end{aligned}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}\{x,y\}$ by 3.3. Comparing coefficients with respect to z:

0) $Q_1^2 + Q_2^2 + Q_3^2 = \alpha_1^2 + \alpha_2^2$

1) $aQ_1 + bQ_2 + cQ_3 + (a^2+b^2+c^2)g = \alpha_1\beta_1 + \alpha_2\beta_2 + \gamma_1 g - \gamma_2 f$

2) $0 = \beta_1^2 + \beta_2^2 - \gamma_1$

and proceeding as in **(A)** above, there exist suitable real numbers $\lambda \neq 0, \mu, u, v, w$ such that

$$\lambda^2(Q_1^2+Q_2^2+Q_3^2) = \big(aQ_1+bQ_2+cQ_3-\mu xy+(a^2+b^2+c^2-\lambda^2)\,\mathrm{In}(g)\big)^2+(ux^2+vy^2+wxy)^2.$$

(i) If $g = x^2 + y^2$, consider

$$G = (3x^2 + 2y^2 - 2z)^2 + (3y^2 + x^2 - z)^2 + \left(\tfrac{21}{10}xy\right)^2,$$

so that

$$\begin{aligned} &\lambda^2\left(10x^4 + \tfrac{2241}{100}x^2y^2 + 13y^4\right) \\ &\quad = \big(-7x^2 - 7y^2 - \mu xy + (5-\lambda^2)(x^2+y^2)\big)^2 + (ux^2+vy^2+wxy)^2 \\ &\quad = \big(\mu xy + (2+\lambda^2)(x^2+y^2)\big)^2 + (ux^2+vy^2+wxy)^2. \end{aligned}$$

for suitable real numbers $\lambda \neq 0, \mu, u, v, w$. Comparing coefficients we get

$x^4)$ $10\lambda^2 = (2+\lambda^2)^2 + u^2$

$y^4)$ $13\lambda^2 = (2+\lambda^2)^2 + v^2$

$x^2y^2)$ $\frac{2241}{100}\lambda^2 = 2(2+\lambda^2)^2 + \mu^2 + 2uv + w^2$

$x^3y)$ $0 = \mu(2+\lambda^2) + uw$

$xy^3)$ $0 = \mu(2+\lambda^2) + vw.$

Substracting the two last equations we deduce $w(u-v)=0$. If $w=0$ then $\mu=0$ and adding the two first equations and substracting the third we obtain $\frac{59}{100}\lambda^2=(u-v)^2$ and combining this with the two first equations we conclude that

$$(u^2+4)(556960000u^4-4850867039u^2+13493610244)=0;$$

but this equation has no real root, and so $w\neq 0$. Thus, $u=v$ and substracting the two first equations we conclude $3\lambda^2=0$, a contradiction.

(ii) If $g=x^2-y^2$, consider

$$G=(3x^2-2y^2-2z)^2+(-3y^2+x^2-z)^2+(3xy)^2.$$

Then

$$\begin{aligned}\lambda^2&\left(10x^4-9x^2y^2+13y^4\right)\\&=\big(-7x^2+7y^2-\mu xy+(5-\lambda^2)(x^2-y^2)\big)^2+(ux^2+vy^2+wxy)^2\\&=\big((2+\lambda^2)(x^2-y^2)+\mu xy\big)^2+(ux^2+vy^2+wxy)^2.\end{aligned}$$

for suitable real numbers $\lambda\neq 0,\mu,u,v,w$. Comparing coefficients we get

$x^4)$ $10\lambda^2=(2+\lambda^2)^2+u^2$

$y^4)$ $13\lambda^2=(2+\lambda^2)^2+v^2$

$x^2y^2)$ $-9\lambda^2=-2(2+\lambda^2)^2+\mu^2+2uv+w^2$

$x^3y)$ $0=\mu(2+\lambda^2)+uw$

$xy^3)$ $0=-\mu(2+\lambda^2)+vw.$

Adding the two last equations we deduce $w(u+v)=0$. If $w=0$ then $\mu=0$; adding the three first equations que deduce $14\lambda^2=(u+v)^2$ and combining this with the two first equations we conclude that

$$(u^2+4)(3136u^4-26015u^2+58564)=0;$$

but this equation has no real root, and so $w\neq 0$. Thus, $u=-v$ and substracting the two first equations we conclude $3\lambda^2=0$, a contradiction.

(iii) If $g=y^2$, consider

$$G=x^4+(-3y^2+z)^2+9x^2y^2.$$

Now

$$\begin{aligned}\lambda^2&\left(x^4+9y^4+9x^2y^2\right)\\&=\big(-3y^2-\mu xy+(1-\lambda^2)y^2\big)^2+(ux^2+vy^2+wxy)^2\\&=\big((2+\lambda^2)y^2+\mu xy\big)^2+(ux^2+vy^2+wxy)^2.\end{aligned}$$

for suitable real numbers $\lambda\neq 0,\mu,u,v,w$. Here we get

$x^4)\quad \lambda^2 = u^2$

$y^4)\quad 9\lambda^2 = (2+\lambda^2)^2 + v^2$

$x^2y^2)\quad 9\lambda^2 = \mu^2 + 2uv + w^2$

$x^3y)\quad 0 = uw$

$xy^3)\quad 0 = -\mu(2+\lambda^2) + vw.$

Since $\lambda \neq 0$, $u \neq 0$ and $w = 0$, hence $\mu = 0$. Thus, $v = \frac{9}{2}u$ and hence $9u^2 = (2+u^2)^2 + (\frac{9}{2}u)^2$ or equivalently $4u^4 + 61u^2 + 16 = 0$, a contradiction. ■

(B2) $\operatorname{rk}(\operatorname{In}(f)) = 1$. To start with, we simplify to the most the equations of X. First, after a change of coordinates, we can suppose that $\operatorname{In}(f) = y^2$ and $\operatorname{In}(g) = ax^2 + bxy + cy^2$. Then

$$\mathcal{J}(X) = (z(z-2g), zf) = (z(z-2g+2cf), zf) = (z(z-2g_1), zf)$$

where $g_1 = g - cf$ has initial form $ax^2 + bxy$, with $a^2 + b^2 \neq 0$. After a suitable change of coordinates of the type $(x, y, z) \mapsto (d_1 x, d_2 y, \pm z)$ we can suppose $\operatorname{In}(g_1) = x^2, x^2 + 2xy$ or xy. If $\operatorname{In}(g_1) = x^2 + 2xy$ then

$$\mathcal{J}(X) = (z(z-2g_1), zf) = (z(z-2(g_1+f)), zf) = (z(z-2g_2), zf)$$

where $g_2 = g_1 - cf$ has initial form $(x+y)^2$ and hence, up to the change $x \mapsto x - y$, we can suppose that $\operatorname{In}(f) = y^2$, $\operatorname{In}(g_2) = x^2$ or xy. Therefore, we have two cases:

(1) $\operatorname{In}(g) = x^2$. Consider the generators of $\mathcal{J}(X)$: $z(z-2(g+f)), zf$. By Morse's Lemma ([Rz1]), after a suitable change of coordinates of the type $\varphi(x, y) = (x + \varphi_1, y + \varphi_2)$, $\omega(\varphi_i) \geq 2$, we can suppose $g + f = x^2 + y^2$. On the other hand, $f = PU$ where $P = y^2 + 2a(x)x^2y + b(x)x^3 \in \mathbb{R}\{x\}[y]$ is a Weierstrass polynomial of degree 2 and $U \in \mathbb{R}\{x, y\}$ is a unit. Thus,

$$\mathcal{J}(X) = \Big(z(z-2(x^2+y^2)), zP\Big) = \Big(z(z-2(x^2+y^2-P)), zP\Big) = \Big(z(z-x^2u(x,y)), zP\Big)$$

where $u(x, y) = 2 - 4a(x)y - 2b(x)x$ is a unit. Moreover, since $\mathcal{J}(X)$ is the ideal of a real surface germ, changing x by $\pm x$, $P = \big(y + a(x)x^2\big)^2 - x^k w(x)$ where $k \geq 3$ and $w \in \mathbb{R}\{x\}$ is a unit with $w(0) > 0$. After the change $y \mapsto y - a(x)x^2$, we have $\mathcal{J}(X) = \Big(z(z - x^2 v(x, y)), z(y^2 - x^k w(x))\Big)$, for a new unit $v(x, y) \in \mathbb{R}\{x, y\}$. Now, after the change $x\sqrt[k]{w(x)} \mapsto x$ we get $\mathcal{J}(X) = \Big(z(z - x^2 v'(x, y)), z(y^2 - x^k)\Big)$, for yet a new unit $v'(x, y) \in \mathbb{R}\{x, y\}$. Finally, after the change $z \mapsto zv'$ we obtain

$$\mathcal{J}(X) = (z(z-x^2), z(y^2 - x^k)), \quad k \geq 3.$$

Now, we find $G \in \mathit{\Sigma}(X) \setminus \mathit{\Sigma}_2(X)$. Again, we have to distinguish two cases:

(a) *k odd.* Take
$$G = (x^2 + xy - z)^2 + (xy)^2 + (y^2)^2.$$

If G were a sum of two squares then there would exist series $\alpha_i, \beta_i \in \mathbb{R}\{x, y\}$, $q_i \in \mathbb{R}\{x, y, z\}$ such that

$$G = (x^2 + xy - z)^2 + (xy)^2 + (y^2)^2 = \\ (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 + q_1 z(z - x^2) + q_2 z(y^2 - x^k). \qquad (*)$$

For $z = 0$, we have

$$(x^2 + xy)^2 + (xy)^2 + (y^2)^2 = \alpha_1^2 + \alpha_2^2, \qquad (**)$$

now, since $\mathbb{C}\{x, y\}$ is an UFD and $G(x, y, 0)$ is a homogeneous polynomial of degree 4, there exist a homogeneous polynomial $P \in \mathbb{C}[x, y]$ of degree 2 and a unit $U \in \mathbb{C}\{x, y\}$ such that $\alpha = \alpha_1 + i\alpha_2 = PU$ and $U\overline{U} = 1$. Thus, if $\beta = \beta_1 + i\beta_2$ we have

$$(\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 = (\alpha + z\beta)\overline{(\alpha + z\beta)} = (PU + z\beta)\overline{U}U\overline{(PU + z\beta)} \\ = (P + z\beta\overline{U})\overline{(P + z\beta\overline{U})} = (P_1 + z(\beta\overline{U})_1)^2 + (P_2 + z(\beta\overline{U})_2)^2.$$

Therefore, we can suppose that α_1, α_2 are homogeneous polynomials of degree 2 and write $\alpha_i = a_i x^2 + b_i xy + c_i y^2$. If $a = a_1 + ia_2, b = b_1 + ib_2, c = c_1 + ic_2$, we have

$$(\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 = (ax^2 + bxy + cy^2 + z\beta)\overline{(ax^2 + bxy + cy^2 + z\beta)} \\ = (ax^2 + bxy + cy^2 + z\beta)\frac{|a|}{a}\frac{|a|}{\overline{a}}\overline{(ax^2 + bxy + cy^2 + z\beta)} \\ = \left(|a|x^2 + \frac{b|a|}{a}xy + \frac{c|a|}{a}y^2 + z\beta\right)\overline{\left(|a|x^2 + \frac{b|a|}{a}xy + \frac{c|a|}{a}y^2 + z\beta\right)}.$$

Hence, we can suppose $a_2 = 0$. Comparing coefficients in $(**)$, we can also suppose $a_1 = b_1 = 1$, and therefore $\alpha_1 = x^2 + xy + c_1 y^2, \alpha_2 = b_2 xy + c_2 y^2$. Now, substituting $z = t^4, x = t^2, y = t^k$ in $(*)$ we deduce

$$2t^{2k+4} + t^{4k} = (t^4(1 + \beta_1(t^2, t^k)) + t^{k+2} + c_1 t^{2k})^2 + (b_2 t^{k+2} + c_2 t^{2k} + t^4\beta_2(t^2, t^k))^2$$

and hence $\omega(1+\beta_1(t^2, t^k)), \omega(\beta_2(t^2, t^k)) \geq k-2$. Since k is odd, $k-2$ is not in the semigroup of the curve germ (t^2, t^k) (see [JP]), and then, in fact, $\omega(1+\beta_1(t^2, t^k)), \omega(\beta_2(t^2, t^k)) \geq k-1$. Therefore, comparing initial forms in the previous equation we deduce $2 = 1 + b_2^2$ and hence we can suppose $b_2 = 1$. Thus, $(x^2+xy)^2+(xy)^2+(y^2)^2 = (x^2+xy+c_1y^2)^2+(xy+c_2y^2)^2$ which transforms into $y^4(c_1^2 + c_2^2 - 1) + 2y^3x(c_1 + c_2) + 2x^2y^2c_1 = 0$, a contradiction.

(b) $k = 2n$ *even*, $n \geq 2$. Take

$$G = (x^2 + xy - z(1 + x^{n-1}))^2 + (xy - zx^{n-1})^2 + (y^2)^2.$$

If G was a sum of two squares then there would exist series $\alpha_i, \beta_i \in \mathbb{R}\{x, y\}$, $q_i \in \mathbb{R}\{x, y, z\}$ such that

$$\begin{aligned} G &= (x^2 + xy - z(1 + x^{n-1}))^2 + (xy - zx^{n-1})^2 + (y^2)^2 \\ &= (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 + q_1 z(z - x^2) + q_2 z(y^2 - x^k) \qquad (*) \end{aligned}$$

As in the previous case, we can take $\alpha_1 = x^2+xy+c_1y^2$, $\alpha_2 = b_2xy+c_2y^2$. Making $z = 0$, we obtain $(x^2+xy)^2+(xy)^2+(y^2)^2 = \alpha_1^2+\alpha_2^2$. Moreover, since $z(y^2-x^k) \in \mathcal{I}(X)$ we can also suppose that $\beta_i = \lambda_i(x) + y\mu_i(x)$ where $\lambda_i, \mu_i \in \mathbb{R}\{x\}$. Now, plugging $x = t, y = t^n, z = t^2$ in the equation $(*)$ we get

$$t^{4n} = (t^2 + t^{n+1} + c_1t^{2n} + t^2\lambda_1(t) + t^{n+2}\mu_1(t))^2 + (b_2t^{n+1} + c_2t^{2n} + t^2\lambda_2(t) + t^{n+2}\mu_2(t))^2.$$

Hence, $\lambda_1(t) = -1 - t^{n-1} + g_1(t)t^n$, $\lambda_2(t) = -b_2t^{n-1} + g_2(t)t^n$ where $g_i \in \mathbb{R}\{\mathtt{t}\}, \omega(g_i) \geq 0$. Finally, plugging $x = t, y = -t^n, z = t^2$ in the equation $(*)$ we get

$$\begin{aligned} &8t^{2n+2} + t^{4n} \\ &\quad = (t^2 - t^{n+1} + c_1t^{2n} + t^2\lambda_1(t) - t^{n+2}\mu_1(t))^2 + (-b_2t^{n+1} + c_2t^{2n} + t^2\lambda_2(t) - t^{n+2}\mu_2(t))^2 \\ &\quad = (-2t^{n+1}+t^{n+2}(g_1(t)-\mu_1(t)+c_1t^{n-2}))^2+(-2b_2t^{n+1}+t^{n+2}(g_2(t)-\mu_2(t)+c_2t^{n-2}))^2. \end{aligned}$$

Comparing initial forms we get $8 = 4 + 4b_2^2$, and we can suppose $b_2 = 1$; proceeding as in the previous case, we get a contradiction. ■

(2) $\mathrm{In}(g) = xy$. By classification of singularities, after a suitable change of coordinates of the type $\varphi(x, y) = (x+\varphi_1, y+\varphi_2)$; $\omega(\varphi_i) \geq 2$, we can suppose $f = xy$. On the other hand, $g = PU$ where $P = y^2+2a(x)x^2y+b(x)x^3 \in \mathbb{R}\{x\}[y]$ is a Weierstrass polynomial of degree 2 and $U \in \mathbb{R}\{x, y\}$ is a unit. Thus, $\mathcal{J}(X) = (z(z - 2xy), zP)$. We recall that since $\mathcal{J}(X)$ is the ideal of a real surface germ (changing x by $\pm x$) $P = \big(y + a(x)x^2\big)^2 - x^kw(x)$ where $k \geq 3$ and $w \in \mathbb{R}\{x\}$ is a unit with $w(0) > 0$. After the change $(y, z) \mapsto (y - a(x)x^2, 2z)$, we have

$$\mathcal{J}(X) = \Big(z(z - x(y - a(x)x^2)), z(y^2 - x^kw(x))\Big).$$

Now, we proceed in the following way: *(1)* if $a(x) = 0$, after the change $(y, z) \mapsto \sqrt{w(x)}(y, z)$ we get

$$\mathcal{J}(X) = (z(z - xy), z(y^2 - x^k)),$$

and, *(2)* if $a(x) \neq 0$ then $a(x) = x^{\ell-2}v(x)$ where $\ell \geq 2$ and $v \in \mathbb{R}\{x\}$ is a unit. We distinguish two cases:

(i) if $k \neq 2\ell$, consider a change of the type $(x, y, z) \mapsto (\lambda(x)x, \mu(x)y, \gamma(x)z)$ where $\lambda, \mu, \gamma \in \mathbb{R}\{x\}$ are units, such that after this change we have

$$\begin{aligned} \mathcal{J}(X) &= \Big(\gamma z(\gamma z - x\lambda(y\mu - v(x\lambda)x^\ell\lambda^\ell)), z\gamma(y^2\mu^2 - x^k\lambda^kw(x\lambda))\Big) \\ &= (z(z - x(y + x^\ell)), z(y^2 - x^k)), \quad k \geq 3, \ell \geq 2 \end{aligned}$$

that is, such that $\mu = -v(x\lambda)\lambda^\ell, \gamma = \lambda\mu, \mu^2 = \lambda^k w(x\lambda)$. This system of equations has a solution if an only if the equation $v(x\lambda)\lambda^{2\ell} = \lambda^k \omega(x\lambda)$ has a solution. But, since $k \neq 2\ell$ we have an equation of the kind $\lambda^n = \varphi(\lambda x)$ where $\varphi \in \mathbb{R}\{t\}$ is a unit, $n \geq 1$, which has a solution by 3.4. Thus, if $\ell \geq k$ then $\mathcal{J}(X) = (z(z - x(y + x^{\ell-k}y)), z(y^2 - x^k)) = (z(z - xy(1 + x^{\ell-k})), z(y^2 - x^k))$ and after the change $z \mapsto z(1 + x^{\ell-k})$ we get $\mathcal{J}(X) = (z(z - xy), z(y^2 - x^k))$.

(ii) if $k = 2\ell$, such a change does not exist; so we proceed as follows. After the change $xw(x) \mapsto x$ we get that $\mathcal{J}(X) = \Big(z(z - x(y - x^\ell u(x))), z(y^2 - x^{2\ell})\Big)$ for certain unit $u \in \mathbb{R}\{x\}$. If $u = \pm 1$ then we get that one of the irreducible 1-dimensional components of X is contained in the 2-dimensional one and therefore X is the union of a plane and a transversal curve, but then our ideal is not $\mathcal{J}(X)$, a contradiction. Therefore, $u \neq \pm 1$ and again we distinguish two cases: *(a)* if $u(0) \neq \pm 1$, after the change $(x, y, z) \mapsto (\frac{x}{\sqrt[\ell]{2}}, y + \frac{x^\ell}{2}, \frac{z}{\sqrt[\ell]{2}})$ we get that

$$\mathcal{J}(X) = \Big(z(z - x(y + x^\ell b(x))), zy(y + x^\ell)\Big),$$

where $b \in \mathbb{R}\{x\}$ is a unit with $b(0) \neq 0, 1$; *b)* if $u(0) = \delta = \pm 1$, we begin by making the change $y \mapsto y - \delta x^\ell$ and we get $\mathcal{J}(X) = \Big(z(z - x(y - x^{l+n}c(x))), z(y^2 + 2y\delta x^\ell)\Big)$ for some unit $c \in \mathbb{R}\{x\}$, $n \geq 1$, then considering a change of the type $(x, y, z) \mapsto (\lambda(x)x, \mu(x)y, \gamma(x)z)$ where $\lambda, \mu, \gamma \in \mathbb{R}\{x\}$ are units and proceeding as before, we come to

$$\mathcal{J}(X) = (z(z - x(y + \delta x^{\ell+r})), zy(y + x^\ell)), \qquad \delta = \pm 1.$$

Now, we find $G \in \Sigma(X) \setminus \Sigma_2(X)$ for the different cases above. Notice that in all of them, $\mathcal{J}(X) = (z(z - x(y + p_1(x))), z(y^2 + p_2(x)y + xp_3(x)))$ where $p_i(x) \in \mathbb{R}\{x\}$ has order ≥ 2 . Let $G \in \Sigma(X)$ be a sum of squares such that $Q(x, y) = G(x, y, 0)$ is a Weierstrass polynomial in the variable y. If G were a sum of two squares then there would exist series $\alpha_i, \beta_i \in \mathbb{R}\{x, y\}$, $q_i \in \mathbb{R}\{x, y, z\}$ such that

$$G = (\alpha_1 + z\beta_1)^2 + (\alpha_2 + z\beta_2)^2 + q_1 z(z - x(y + p_1(x))) + q_2 z(y^2 + p_2(x)y + xp_3(x)); \quad (*)$$

moreover, since $z(y^2 + p_2(x)y + xp_3(x)) \in \mathcal{I}(X)$ we can suppose that $\beta_i = \lambda_i(x) + y\mu_i(x)$ where $\lambda_i, \mu_i \in \mathbb{R}\{x\}$. Making $z = 0$ in the expression above, we get

$$Q(x, y) = \alpha_1^2 + \alpha_2^2,$$

and using the fact that $\mathbb{C}\{x\}[y]$ is an UFD, that $Q(x, y) = G(x, y, 0)$ is a Weierstrass polynomial with respect to y, the uniqueness in the Weierstrass Preparation Theorem and proceeding similarly to ***(B2.1)*** we can suppose $\alpha = \alpha_1 + i\alpha_2 = (y - \xi_1(x))(y - \xi_2(x))$ where $\xi_1(x), \xi_2(x) \in \mathbb{C}\{x\}$ are two non-conjugated roots of $Q(x, y)$. Now, we proceed with the different cases above:

(a) If $\mathcal{J}(X) = (z(z - xy), z(y^2 - x^k))$, we take

$$G = (y^2 - x^k)^2 + (x^k - \varepsilon_k z x^{k/2-1})^2 + (xy - z)^2,$$

where $\varepsilon_k = 0$ if k is odd and 1 otherwise. Let

$$\begin{aligned}\eta_1 &= x^{k-1}g_1(x) + i(x + x^{k-1}g_2(x))\\ \eta_2 &= x^{k-1}g_1(x) - i(x + x^{k-1}g_2(x))\\ \eta_3 &= x^{2k-3}h_1 + i(\sqrt{2}x^{k-1} + x^{2k-3}h_2)\\ \eta_4 &= x^{2k-3}h_1 - i(\sqrt{2}x^{k-1} + x^{2k-3}h_2)\end{aligned}$$

be the roots of Q in $\mathbb{C}\{x\}$, where $g_i, h_i \in \mathbb{R}\{x\}$. We can suppose

$$\alpha = \alpha_1(x,y) + i\alpha_2(x,y) = y^2 \pm \sqrt{2}x^k + yx^{k-1}\varphi_1(x) + x^{2k-2}\psi_1(x) \\ + i(-xy + yx^{k-1}\varphi_2(x) + x^{2k-2}\psi_2(x))$$

for suitable series $\varphi_i, \psi_i \in \mathbb{R}\{x\}$. Now, we have to distinguish two cases: *(1)* If k is odd, substituting $z = xy$, $x = t^2$, $y = t^k$ in the expression $(*)$ (adapted to this case), we obtain that

$$t^{4k} = ((1 \pm \sqrt{2})t^{2k} + t^{3k-2}\gamma_1(t) + t^{k+2}(\lambda_1(t^2) + t^k\mu_1(t^2)))^2 \\ + (-t^{k+2} + t^{3k-2}\gamma_2(t) + t^{k+2}(\lambda_2(t^2) + t^k\mu_2(t^2)))^2$$

where $\gamma_i \in \mathbb{R}\{t\}$. Hence, since k is odd, from the previous equation we conclude $1 = (1 \pm \sqrt{2})^2$, a contradiction. *(2)* If $k = 2n$ is even, substituting $z = xy$, $x = t$, $y = t^n$ in the expression $(*)$, we obtain that

$$0 = ((1 \pm \sqrt{2})t^{2n} + t^{3n-2}\gamma_1(t) + t^{n+1}(\lambda_1(t) + t^n\mu_1(t)))^2 \\ + (-t^{n+1} + t^{3n-2}\gamma_2(t) + t^{n+1}(\lambda_2(t) + t^n\mu_2(t)))^2$$

where $\gamma_i \in \mathbb{R}\{t\}$. Thus, we deduce that $\lambda_1 = -(1 \pm \sqrt{2})t^{n-1} + \cdots$, $\lambda_2 = 1 + \lambda_{2,n}t^n + \cdots$. Now, substituting $z = xy$, $x = t$, $y = -t^n$ in the expression $(*)$, we obtain that

$$4t^{2k} = ((1 \pm \sqrt{2})t^{2n} + t^{3n-2}\gamma_1'(t) - t^{n+1}(\lambda_1(t) - t^n\mu_1(t)))^2 \\ + (t^{n+1} + t^{3n-2}\gamma_2'(t) - t^{n+1}(\lambda_2(t) - t^n\mu_2(t)))^2$$

where $\gamma_i' \in \mathbb{R}\{t\}$. Hence, putting all together, we deduce that $4 = 4(1 \pm \sqrt{2})^2$, a contradiction. ■

(b) If $\mathcal{J}(X) = (z(z - x(y + x^\ell)), z(y^2 - x^k))$, $\; 2 \leq \ell < k, 3 \leq k \neq 2\ell$ we have to distinguish four different situations:

i) If k is odd and $2\ell > k$, we take $G = y^4 + (x^{k-l-1}(xy - z))^2 + x^{2k}$. Let

$$\begin{aligned}\eta_1 &= x^{5\ell-2k}g_1 + i(x^\ell + x^{5\ell-2k}g_2)\\ \eta_2 &= x^{5\ell-2k}g_1 - i(x^\ell + x^{5\ell-2k}g_2)\\ \eta_3 &= x^{3\ell-k}h_1 + i(x^{k-l} + x^{3\ell-k}h_2)\\ \eta_4 &= x^{3\ell-k}h_1 - i(x^{k-l} + x^{3\ell-k}h_2)\end{aligned}$$

be the roots of Q in $\mathbb{C}\{x\}$, where $g_i, h_i \in \mathbb{R}\{x\}$. Therefore, if $\beta = \beta_1 + i\beta_2$ and $\varepsilon = \pm 1$ substituting $z = xy + x^{\ell+1}, x = t^2, y = t^k$ in the expression $(*)$, we get

$$\begin{aligned} 3t^{4k} &\equiv G(t^2, t^k, t^{k+2} + t^{2\ell+2}) = |(\alpha + z\beta)(t^2, t^k, t^{k+2} + t^{2\ell+2})|^2 \\ &\equiv |(t^k - it^{2\ell})(t^k + \varepsilon i t^{2k-2\ell}) + (t^{k+2} + t^{2\ell+2})(\lambda(t^2) + t^k\mu(t^2))|^2 \\ &\equiv |(1+\varepsilon)t^{2k} + i(\varepsilon t^{3k-2\ell} - t^{k+2\ell}) + (t^{k+2} + t^{2\ell+2})\lambda(t^2)| \mod (t^{4k+1}). \end{aligned}$$

Since k is odd we deduce, comparing orders and initial forms, that $\lambda(t^2) = -i\varepsilon t^{2(k-\ell)-2} + \cdots$ and thus $3 = (1+\varepsilon)^2 + (\varepsilon)^2$, a contradiction.

ii) If k is odd and $2\ell < k$, we take $G = (y^2 + x^{\ell+1} - z)^2 + (xy - x^{\ell+1} + z)^2 + (x^{\ell+1} - z)^2$. Let

$$\begin{aligned} \eta_1 &= x^\ell g_1 + i(x + x^\ell g_2) \\ \eta_2 &= x^\ell g_1 - i(x + x^\ell g_2) \\ \eta_3 &= -x^\ell + x^{2\ell-1}h_1 + i(\sqrt{2}x^\ell + x^{2\ell-1}h_2) \\ \eta_4 &= -x^\ell + x^{2\ell-1}h_1 - i(\sqrt{2}x^\ell + x^{2\ell-1}h_2) \end{aligned}$$

be the roots of Q in $\mathbb{C}\{x\}$, where $g_i, h_i \in \mathbb{R}\{x\}$. Therefore, if $\beta = \beta_1 + i\beta_2$ and $\varepsilon = \pm 1$ substituting $z = x^{\ell+1} + xy, x = t^2, y = t^k$ in the expression $(*)$, we get

$$\begin{aligned} 6t^{2k+4} - 2t^{3k+2} + t^{4k} &= G(t^2, t^k, t^{2\ell+2} + t^{k+2}) = |(\alpha + z\beta)(t^2, t^k, t^{k+1} + t^{2\ell+2})|^2 \\ = |(t^k - t^{2\ell}g_1(t^2) - i(t^2 + t^{2\ell}g_2(t^2)))&(t^k + t^{2\ell} - t^{4\ell-2}h_1(t^2) + \varepsilon i(\sqrt{2}t^{2\ell} + t^{4\ell-2}h_2(t^2))) \\ &+ (t^{2\ell+2} + t^{k+2})(\lambda(t^2) + t^k\mu(t^2))|^2, \end{aligned}$$

Since $k > 2\ell$, comparing initial forms, we deduce that $\lambda(0) = i - \varepsilon\sqrt{2}$. Moreover, using also that k is odd and $\ell \geq 2$ we conclude that $6 = |-i + \lambda(0)|^2 = 2$, a contradiction.

iii) If $k = 2n$ is even and $\ell > n$, let us take

$$G = \left(y^2 - \frac{zx^{n-1}}{1 + x^{\ell-n}}\right)^2 + \left(x^{k-\ell-1}(xy - z) + \frac{zx^{n-1}}{1 + x^{\ell-n}}\right)^2 + \left(x^k - \frac{zx^{n-1}}{1 + x^{\ell-n}}\right)^2.$$

Let $\eta_1, \eta_2, \eta_3, \eta_4$ be the roots of Q in $\mathbb{C}\{x\}$ which are the same obtained in *a)*. Therefore, if $\beta = \beta_1 + i\beta_2$ and $\varepsilon = \pm 1$ substituting $z = xy + x^{\ell+1}, x = t, y = t^n$ in the expression $(*)$, we get

$$\begin{aligned} 0 &\equiv G(t, t^n, t^{n+1} + t^{\ell+1}) = |(\alpha + z\beta)(t, t^n, t^{n+1} + t^{\ell+1})|^2 \\ &\equiv |(t^n - it^\ell)(t^n + \varepsilon i t^{2n-\ell}) + (t^{n+1} + t^{\ell+1})(\lambda(t) + t^n\mu(t))|^2 \\ &\equiv |(1+\varepsilon)t^{2n} + i(\varepsilon t^{3n-\ell} - t^{n+\ell}) + (t^{n+1} + t^{\ell+1})(\lambda(t) + t^n\mu(t))| \mod (t^{4n+1}) \end{aligned}$$

and, comparing orders and initial forms, we deduce that $\lambda(t) = -\varepsilon t^{2n-\ell-1} - (1+\varepsilon - i\varepsilon)t^{n-1} + \cdots$. Now, substituting $z = xy + x^{\ell+1}, x = t, y = -t^n$ we get

$$\begin{aligned}
12t^{4n} &\equiv G(t, -t^n, t^{n+1} + t^{\ell+1}) = |(\alpha + z\beta)(t, -t^n, t^{n+1} + t^{\ell+1})|^2 \\
&\equiv |(-t^n - it^\ell)(-t^n + \varepsilon i t^{2n-\ell}) + (-t^{n+1} + t^{\ell+1})(\lambda(t) - t^n\mu(t))|^2 \\
&\equiv |(1+\varepsilon)t^{2n} + i(-\varepsilon t^{3n-\ell} + t^{n+\ell}) + (-t^{n+1} + t^{\ell+1})(\lambda(t) + t^n\mu(t))| \\
&\equiv |(2 + 2\varepsilon - \varepsilon i)t^{2n}|^2 \mod (t^{4n+1})
\end{aligned}$$

which is impossible.

iv) If $k = 2n$ is even and $\ell < n$, we take

$$G = \left(y^2 + x^{2\ell} - z\frac{x^{\ell-1} + yx^{n-\ell-1}}{1 + x^{n-\ell}}\right)^2 + \left(x^\ell y - z\frac{x^{n-1}}{1 + x^{n-\ell}}\right)^2 + (x^{2\ell} + x^{n+\ell} - zx^{\ell-1})^2.$$

Let

$$\begin{aligned}
\eta_1 &= x^{2n-\ell}g_1 + i(x^\ell + x^n + x^{2n-\ell}g_2) \\
\eta_2 &= x^{2n-\ell}g_1 - i(x^\ell + x^n + x^{2n-\ell}g_2) \\
\eta_3 &= x^{2n-\ell}h_1 + i(\sqrt{2}x^\ell - \sqrt{2}/2x^n + x^{2n-\ell}h_2) \\
\eta_4 &= x^{2n-\ell}h_1 - i(\sqrt{2}x^\ell - \sqrt{2}/2x^n + x^{2n-\ell}h_2)
\end{aligned}$$

be the roots of Q in $\mathbb{C}\{x\}$, where $g_i, h_i \in \mathbb{R}\{x\}$. Therefore, if $\beta = \beta_1 + i\beta_2$ and $\varepsilon = \pm 1$ substituting $z = x^{\ell+1} + xy, x = t, y = t^n$ in the expression $(*)$, we get

$$\begin{aligned}
0 &\equiv G(t, t^n, t^{\ell+1} + t^{n+1}) = |(\alpha + z\beta)(t, t^n, t^{\ell+1} + t^{n+1})|^2 \\
&\equiv |(t^n - i(t^l + t^n))(t^n + \varepsilon i(\sqrt{2}t^\ell - \sqrt{2}/2t^n)) + (t^{\ell+1} + t^{n+1})(\lambda(t) + t^n\mu(t))|^2 \\
&\equiv |\varepsilon\sqrt{2}t^{2\ell} + (\varepsilon\sqrt{2}/2 + i(\varepsilon\sqrt{2} - 1))t^{n+l} + (1 - \varepsilon\sqrt{2}/2 - i(1 + \varepsilon\sqrt{2}/2))t^{2n} \\
&\qquad + (t^{\ell+1} + t^{n+1})(\lambda(t) + t^n\mu(t))|^2 \mod (t^{2\ell+2n+1})
\end{aligned}$$

Hence, we get that $\lambda(t) = -\varepsilon\sqrt{2}t^{\ell-1} + t^{n-1}(\varepsilon\sqrt{2}/2 + i(1 - \varepsilon\sqrt{2})) + \cdots$. Now, substituting $z = x^{\ell+1} + xy, x = t, y = -t^n$ we obtain

$$\begin{aligned}
12t^{2\ell+2n} &\equiv G(t, -t^n, t^{\ell+1} - t^{n+1}) = |(\alpha + z\beta)(t, -t^n, t^{\ell+1} - t^{n+1})|^2 \\
&\equiv |(-t^n - i(t^l + t^n))(-t^n + \varepsilon i(\sqrt{2}t^\ell - \sqrt{2}/2t^n)) + (t^{\ell+1} - t^{n+1})(\lambda(t) + t^n\mu(t))|^2 \\
&\equiv |2(\varepsilon\sqrt{2} + i(1 - \varepsilon\sqrt{2}))t^{n+\ell}|^2 \mod (t^{2\ell+2n+1})
\end{aligned}$$

which is impossible. ■

(c) If $\mathcal{J}(X) = \Big(z(z - x(y + x^\ell b(x))), zy(y + x^\ell)\Big)$, $2 \leq \ell$ and $b \in \mathbb{R}\{x\}$ with $b(0) \neq 0, 1$ we take

$$G = (y^2 - x^{2\ell})^2 + 2\left(x^\ell y - \frac{zx^{\ell-1}}{1 - b(x)}\right)^2 + 3\left(x^{2\ell} + \frac{zx^{\ell-1}}{1 - b(x)}\right)^2.$$

Let $\eta_1=(1+i)x^\ell, \eta_2=(1-i)x^\ell, \eta_3=(-1+i)x^\ell, \eta_4=(-1-i)x^\ell$ be the roots of Q in $\mathbb{C}\{x\}$. Therefore, if $\beta=\beta_1+i\beta_2$ and $\varepsilon=\pm 1$ substituting $z=x(y+x^\ell b(x))), x=t, y=-t^\ell$ in the expression $(*)$, we get

$$\begin{aligned}0&=G(t,-t^\ell,-t^{\ell+1}(1-b(t)))=|(\alpha+z\beta)(t,-t^\ell,-t^{\ell+1}(1-b(t)))|^2\\&=|(-t^\ell-(1+i)t^\ell)(-t^\ell-(-1+\varepsilon i)t^\ell)-t^{\ell+1}(1-b(t))(\lambda(t)-t^\ell\mu(t))|^2\\&=|t^{2\ell}\varepsilon(2i-1)-t^{\ell+1}(1-b(t))(\lambda(t)-t^\ell\mu(t))|^2\end{aligned}$$

Hence, we get that $\lambda(t)=t^{\ell-1}\varepsilon(2i-1)/(1-b(0))+\cdots$. Now, substituting $z=x(y+x^\ell b(x))), x=t, y=0$ in the expression $(*)$, we obtain

$$\begin{aligned}\frac{t^{4\ell}(4-2b(t)+3b(t)^2)}{(b(t)-1)^2}&=G(t,0,t^{\ell+1}b(t))=|(\alpha+z\beta)(t,0,t^{\ell+1}b(t))|^2\\&=|(-(1+i)t^\ell)(-(-1+\varepsilon i)t^\ell)+t^{\ell+1}b(t)\lambda(t)|^2\\&=|t^{2\ell}(-1-\varepsilon+(-1+\varepsilon)i)+t^{\ell+1}b(t)\lambda(t)|^2\end{aligned}$$

and therefore, comparing initial forms, we conclude

$$\begin{aligned}&4-2b(0)+3b(0)^2=|(-1-\varepsilon+(-1+\varepsilon)i)(1-b(0))+b(0)\varepsilon(2i-1)|^2\\&=(1+\varepsilon-b(0))^2+(-1+\varepsilon+b(0)+\varepsilon b(0))^2=4-2b(0)-2\varepsilon b(0)+3b(0)^2+2\varepsilon b(0)^2.\end{aligned}$$

Hence, $b(0)-b(0)^2=0$, which is impossible. ■

(d) If $\mathcal{J}(X)=(z(z-x(y+\delta x^{\ell+n})), zy(y+x^\ell)),\quad 2\le\ell, 1\le n$ and $\delta=\pm1$ we take

$$G=(y^2)^2+2(x^\ell y)^2+(x^{2\ell+n}-\delta z x^{\ell-1})^2.$$

Let

$$\begin{aligned}\eta_1&=x^{\ell+2n}g_1+i(x^\ell\sqrt{2}+x^{\ell+2n}g_2)\\\eta_2&=x^{\ell+2n}g_1-i(x^\ell\sqrt{2}+x^{\ell+2n}g_2)\\\eta_3&=x^{\ell+3n}h_1+i(\sqrt{2}/2\ x^{\ell+n}+x^{\ell+3n}h_2)\\\eta_4&=x^{\ell+3n}h_1-i(\sqrt{2}/2\ x^{\ell+n}+x^{\ell+3n}h_2)\end{aligned}$$

be the roots of Q in $\mathbb{C}\{x\}$, where $g_i,h_i\in\mathbb{R}\{x\}$. Therefore, if $\beta=\beta_1+i\beta_2$ and $\varepsilon=\pm1$ substituting $z=x(y+\delta x^{\ell+n}), x=t, y=0$ in the expression $(*)$, we get

$$\begin{aligned}0\equiv G(t,0,\delta t^{\ell+n+1})&=|(\alpha+z\beta)(t,0,\delta t^{\ell+n+1})|^2\\&\equiv|i(t^\ell\sqrt{2})\varepsilon i(\sqrt{2}/2\ t^{\ell+n})+\delta t^{\ell+n+1}\lambda(t)|^2\\&\equiv|-\varepsilon t^{2\ell+n}+\delta t^{\ell+n+1}\lambda(t)|^2\mod(t^{4\ell+2n+1})\end{aligned}$$

Hence, we get that $\lambda(t) = \varepsilon\delta t^{\ell-1} + \cdots$. Now, substituting $z = x(y + \delta x^{\ell+n}), x = t, y = -t^\ell$ we obtain

$$\begin{aligned} 4t^{4\ell} &\equiv G(t, -t^\ell, -t^{\ell+1} + \delta t^{\ell+n+1}) = |(\alpha + z\beta)(t, -t^\ell, -t^{\ell+1} + \delta t^{\ell+n+1})|^2 \\ &\equiv |(-t^\ell - it^\ell\sqrt{2})(-t^\ell - \varepsilon i\sqrt{2}/2\; t^{\ell+n}) - (t^{\ell+1} - \delta t^{\ell+n+1})(\lambda(t) - t^\ell\mu(t))|^2 \\ &\equiv |(1 + \sqrt{2}i - \varepsilon\delta)t^{2\ell}|^2 \mod (t^{4\ell+1}) \end{aligned}$$

and therefore, $2 = (1 - \varepsilon\delta)^2$, a contradiction. ■

(C) $\omega(f) = 2$, $\omega(g) = 2$ *and their initial forms are linearly dependent.* Then, we can write $\mathrm{In}(g) = \lambda\,\mathrm{In}(f)$ and therefore

$$\begin{aligned} \mathcal{J}(X) &= (z(z - 2g(x,y)), zf(x,y)) \\ &= (z(z - 2(g(x,y) - \lambda f(x,y))), zf(x,y)) = (z(z - 2h(x,y)), zf(x,y)). \end{aligned}$$

where $h = g(x,y) - \lambda f(x,y)$ has order ≥ 3, and we are in the following case. ■

(D) $\omega(f) = 2$, $\omega(g) \geq 3$. Again, after a change and a little computation, we can assume that $f = x^2 - y^k$ for $k \geq 2$ and $g = y^{\ell+1}(xa(y) + b(y))$ for some $\ell \geq 1$ and $a, b \in \mathbb{R}\{y\}$ such that $\omega(a^2 + b^2) = 0$. Let $\psi = \sum_{i=1}^3 (a_i + zb_i)^2$ ($a_i, b_i \in \mathbb{R}\{x,y\}$) be a sum of three squares which is not a sum of squares in the ring of the set germ X' given by the equations $zf = 0, z(z + (xa(y) + b(y))y) = 0$ (it exist as we have seen in **(B)**). Then the function germ in $\mathcal{O}(X)$ given by $\varphi = \sum_{i=1}^3 (a_i y^\ell + zb_i)^2$ is in $\Sigma_3(X) \setminus \Sigma_2(X)$. Indeed, if $\varphi \in \Sigma_2(X)$ then

0) $\sum_i a_i^2 y^{2\ell} = \alpha_1^2 + \alpha_2^2$

1) $\sum_i a_i y^\ell b_i + \sum_i b_i^2 y^\ell (xa(y) + b(y))y = y^\ell(xa(y) + b(y))y(\beta_1^2 + \beta_2^2) + \gamma_2 f + \alpha_1\beta_1 + \alpha_2\beta_2$.

for certain $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}\{x,y\}$. From *0)* we deduce that $y^\ell | \alpha_j$, and hence, from *1)* , $y^\ell | \gamma_2$. Therefore we can write $\alpha_j = y^\ell \alpha_j'$, $\gamma_2 = y^\ell \gamma_2'$ and the we have

0) $\sum_i a_i^2 = (\alpha_1')^2 + (\alpha_2')^2$

1) $\sum_i a_i b_i + \sum_i b_i^2 (xa(y) + b(y))y = (xa(y) + b(y))y(\beta_1^2 + \beta_2^2) + \gamma_2' f + \alpha_1'\beta_1 + \alpha_2'\beta_2$.

which means that $\psi \in \Sigma_2(X')$, a contradiction. Then φ is not a sum of two squares in $\mathcal{O}(X)$, and we are done. ■

4 Examples in higher embedding dimension

In this section we discuss the examples X_n (Veronese cones), Y_n (generalized Whitney's umbrellas) and Z_n. We begin by proving that:

Theorem 4.1 $\mathcal{P}(X_n) = \Sigma_2(X_n)$.

First, we show:

Lemma 4.2 *For every* $f \in \mathbb{R}\{x_0, x_1, \ldots, x_n\}$ *there exist* $f_0, f_1, \ldots, f_{n-1} \in \mathbb{R}\{x_n\}$ *and* $g \in \mathbb{R}\{x_0, x_1, \ldots, x_n\}$ *such that*

$$f = f_0(x_n) + \sum_{i=1}^{n-1} f_i(x_n)x_i + x_0 g \qquad \mod \mathcal{J}(X_n).$$

Proof. We consider for every $\nu = (\nu_1, \ldots, \nu_{n-1})$ the homogeneous polynomial

$$G_\nu = x_1^{\nu_1} \cdots x_n^{\nu_{n-1}} - x_0^{d-1-k} x_n^k x_i$$

where $d = |\nu|$, $0 \leq i < n$ and $\sum_{j=1}^{n-1} j\nu_j = nk + i$. Since

$$G_\nu \circ \gamma = \prod_{j=1}^{n-1} (z^{n-j}w^j)^{\nu_j} - z^{n(d-1-k)} w^{nk} z^{n-i} w^i = z^{nd-nk-i} w^{nk+i} - z^{n(d-k)-i} w^{nk+i} = 0$$

we see that $G_\nu \in \mathcal{J}(X_n)$; for $\nu = (0, \ldots, 1, \ldots, 0)$ we get $x_i^n - x_n^i x_0^{n-i} \in \mathcal{J}(X_n)$. Therefore, we divide $f \in \mathbb{R}\{x_0, \ldots, x_n\}$ succesively by these polynomials $x_i^n - x_n^i x_0^{n-i}$ until we obtain

$$f = \sum_{0 \leq \nu_1, \ldots, \nu_{n-1} < n} a_\nu(x_0, x_n) x_1^{\nu_1} \cdots x_{n-1}^{\nu_{n-1}} \qquad \mod \mathcal{J}(X_n).$$

Furthermore, $G_\nu \in \mathcal{J}(X_n)$ means $x_1^{\nu_1} \cdots x_n^{\nu_{n-1}} = x_0^{d-1-k} x_n^k x_i \quad \mod \mathcal{J}(X_n)$, and we obtain $b_0, b_1, \ldots, b_{n-1} \in \mathbb{R}\{x_0, x_n\}$ such that

$$f = b_0(x_0, x_n) + \sum_{i=1}^{n-1} b_i(x_0, x_n) x_i \qquad \mod \mathcal{J}(X_n).$$

Finally, since $b_i(x_0, x_n) = f_i(x_n) + x_0 g_i(x_0, x_n)$, there exists $g \in \mathbb{R}\{x_0, x_1, \ldots, x_n\}$ such that

$$f = f_0(x_n) + \sum_{i=1}^{n-1} f_i(x_n)x_i + x_0 g \qquad \mod \mathcal{J}(X_n).$$

■

To prove that $\mathcal{P}(X_n) = \Sigma_2(X_n)$ we need the following polynomial reduction lemma.

Lemma 4.3 *For every $f \in \mathcal{P}(X_n)$ and every $k \geq 1$ there exists a polynomial f_k positive semidefinite on the algebraic surface S_n given by the same equations as X_n such that $\omega(f - f_k) > k$.*

Proof. We parametrize S_n as follows. If n is odd, we take the complex parametrization γ, which maps $\mathbb{R}^2$ over S_n. We write $\gamma_+ = \gamma|_{\mathbb{R}^2}$. If n is even γ_+ only parametrizes $S_n \cap \{x_0 \geq 0\}$ and we must use $\gamma_- = -\gamma_+$ to parametrize $S_n \cap \{x_0 < 0\}$.

Now, choose $k \geq 1$ and $f \in \mathcal{P}(X_n)$ and set $g_k = f + (x_0^2 + x_1^2 + \ldots + x_n^2)^k$. We claim that

$$g_k + (x_0, x_1, \ldots, x_n)^r \subset \mathcal{P}^+(X_n).$$

for $r \geq 2k$ big enough.

Indeed, the germs $g_k \circ \gamma_+$ and $g_k \circ \gamma_-$ are positive definite in $\mathbb{R}^2$. In ([Fe2, 3.1]) we showed that if a function germ g is positive semidefinite in a semianalytic germ Z of $\mathbb{R}^2$ there exits an integer r such that all function germs in $g + \mathfrak{m}_2^r$ are also postive semidefinite in Z. Therefore, in our case, there exist $r \geq 2k$ such that

$$g_k \circ \gamma_+ + (s,t)^{rn}, \qquad g_k \circ \gamma_- + (s,t)^{rn} \subset \mathcal{P}^+(\mathbb{R}^2).$$

from which the claim follows.

We consider now the $(r-1)$ –jet h_k of g_k, which as we have seen is positive definite in X_n. Therefore, there exists $\varepsilon > 0$ such that h_k is ≥ 0 in $S_n \cap B_\epsilon(0)$. On the other hand, if $y \in S_n \cap \mathbb{R}^{n+1} \setminus B_\varepsilon(0)$, then $\|y\| \geq \varepsilon$, and we deduce

$$\begin{aligned} |h_k(y)| = \Big| \sum_{0 \leq |\nu| \leq r-1} a_\nu y^\nu \Big| &\leq \sum_{0 \leq |\nu| \leq r-1} |a_\nu| |y_0^{\nu_0}| |y_1^{\nu_1}| \ldots |Z_n^{\nu_n}| \\ &\leq \sum_{0 \leq |\nu| \leq r-1} |a_\nu| \|y\|^{|\nu|} \leq \sum_{0 \leq |\nu| \leq r-1} \frac{|a_\nu|}{\varepsilon^{2r-|\nu|}} \|y\|^{2r} \leq M \|y\|^{2r} \end{aligned}$$

Hence the polynomial $f_k = h_k + M(x_0^2 + x_1^2 + \ldots + x_n^2)^r$ is ≥ 0 on S_n and $\omega(f - f_k) > k$. ■

Now we proceed with the

Proof of Theorem 4.1. From the previous lemmas and the M. Artin's Approximation Theorem, it suffices to prove that every polynomial f which is positive semidefinite on S_n is a sum of two squares of analytic function germs on X_n. To that end, we consider the biregular equivalence

$$\begin{aligned} \phi_n : \mathbb{R}^2 \setminus \{x_0 = 0\} &\to S_n \setminus \{x_0 = 0\} \\ (x_0, x_1) &\mapsto \Big(x_0, x_1, \frac{x_1^2}{x_0}, \ldots, \frac{x_1^k}{x_0^{k-1}}, \ldots, \frac{x_1^n}{x_0^{n-1}} \Big) \end{aligned}$$

whose inverse π is the obvious projection. Now, let

$$g = f \circ \phi_n = f\Big(x_0, x_1, \frac{x_1^2}{x_0}, \dots, \frac{x_1^k}{x_{k-1}}, \dots, \frac{x_1^n}{x_0^{n-1}}\Big) = \frac{P(x_0, x_1)}{x_0^{2r}},$$

where $r \geq 0$, and $P \in \mathbb{R}[x_0, x_1]$ is ≥ 0 on $x_0 \neq 0$, hence on $\mathbb{R}^2$. Since $\mathcal{P} = \Sigma_2$ in $\mathbb{R}\{x_1, x_2\}$, we have

$$x_0^{2r} g = P = a^2 + b^2, \quad a, b \in \mathbb{R}\{x_0, x_1\}.$$

Therefore, composing with π we obtain

$$x_0^{2r} f = (a^2 + b^2) \qquad \mathrm{mod}\ \mathcal{J}(X_n). \qquad (*)$$

In view of 4.2, there exist power series $a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1} \in \mathbb{R}\{x_n\}$ and $\alpha, \beta \in \mathbb{R}\{x_0, x_1, \dots, x_n\}$ such that

$$a \equiv a_0(x_n) + \sum_{i=1}^{n-1} a_i(x_n) x_i + x_0 \alpha \qquad \mathrm{mod}\ \mathcal{J}(X_n),$$
$$b \equiv b_0(x_n) + \sum_{i=1}^{n-1} b_i(x_n) x_i + x_0 \beta \qquad \mathrm{mod}\ \mathcal{J}(X_n),$$

and hence,

$$x_0^{2r} f = \left(a_0(x_n) + \sum_{i=1}^{n-1} a_i(x_n) x_i + x_0\alpha\right)^2 + \left(b_0(x_n) + \sum_{i=1}^{n-1} b_i(x_n) x_i + x_0\beta\right)^2 \mathrm{mod}\ \mathcal{J}(X_n).$$

Substituting γ_+ we get

$$s^{2rn}(f \circ \gamma_+) = \left(a_0(t^n) + \sum_{i=1}^{n-1} a_i(t^n) s^{n-i} t^i + s^n(\alpha \circ \gamma_+)\right)^2 + \left(b_0(t^n) + \sum_{i=1}^{n-1} b_i(t^n) s^{n-i} t^i + s^n(\beta \circ \gamma_+)\right)^2$$

and counting orders respect to s

$$\mathrm{ord}_s\left(a_0(t^n) + \sum_{i=1}^{n-1} a_i(t^n) s^{n-i} t^i + s^n(\alpha \circ \gamma_+)\right) \geq rn,$$
$$\mathrm{ord}_s\left(b_0(t^n) + \sum_{i=1}^{n-1} b_i(t^n) s^{n-i} t^i + s^n(\beta \circ \gamma_+)\right) \geq rn.$$

Thus we deduce that $a_i(t^n), b_i(t^n) = 0$ for $0 \le i \le n-1$ and consequently, $a_i, b_i = 0$ for $0 \le i \le n-1$. Therefore, $x_0^{2r} f = x_0^2(\alpha^2 + \beta^2) \mod \mathcal{J}(X_n)$ and since $x_0 \notin \mathcal{J}(X_n)$ and this ideal is prime, we conclude

$$x_0^{2r-2} f = (\alpha^2 + \beta^2) \qquad \mod \mathcal{J}(X_n).$$

Whence, we can begin again the argument from $(*)$ and at the end we will obtain $f \in \Sigma_2(X_n)$, as wanted. ■

Next we turn to Whitney's umbrellas:

Theorem 4.4 $\mathcal{P}(Y_n) = \Sigma_2(Y_n)$.

Proof. The parametrization

$$\varphi_n : (s,t) \mapsto (s, st, \dots, st^{n-1}, t^n) = (x_0, x_1, \dots, x_{n-1}, x_n).$$

induces a homomorphism of rings

$$\begin{array}{rccc} \varphi_n^* : & \mathcal{O}(Z_n) & \to & \mathbb{R}\{s,t\} \\ & f & \mapsto & f \circ \varphi_n. \end{array}$$

which is finite, injective and $(s)\,\mathbb{R}\{s,t\} \subset \operatorname{im} \psi_n$ (the last remark, because $s^i t^j = s^{i-1}(st^r)t^{nq} = \psi_n(x_0^{i-1} x_r x_n)$ where $j = nq + r$, $0 \le r < n$).

Let $f \in \mathcal{P}(Y_n)$ and consider $f \circ \varphi_n$. Since f is a psd in $\mathcal{O}(Y_n)$ then $f \circ \varphi_n$ is psd in $\mathbb{R}\{s,t\}$ and there exist $\alpha_1, \alpha_2, \in \mathbb{R}\{s,t\}$ and $\beta_1, \beta_2 \in \mathbb{R}\{t\}$ such that

$$f \circ \varphi_n \equiv (\alpha_1 s + \beta_1)^2 + (\alpha_2 s + \beta_2)^2.$$

It is clear that $C_n = Z_n \cap \{x_0 = 0\}$ is the line $x_0 = 0, \dots, x_{n-1} = 0$, which has Pythagoras number 1. Setting $g = f|_{C_n} \in \mathcal{P}(C_n)$, there exists $g_1 \in \mathbb{R}\{x_n\}$ such that $g \equiv g_1^2 \mod \mathcal{J}(C_n)$. Thus, if $\gamma_1 = g_1 \circ \varphi_n(0,t)$, $i = 1, 2$, we deduce

$$\beta_1^2 + \beta_2^2 = f \circ \varphi_n(0,t) = f|_{C_n} \circ \varphi_n(0,t) = g_1^2 \circ \varphi_n(0,t) = \gamma_1^2,$$

and $\dfrac{\gamma_1}{\beta_1 + i\beta_2}, \dfrac{\gamma_1}{\beta_1 - i\beta_2}$ are two units in $\mathbb{C}\{t\}$ whose product is 1. Hence,

$$\begin{aligned} &(\alpha_1 s + \beta_1)^2 + (\alpha_2 s + \beta_2)^2 \\ &\qquad = (\alpha_1 s + i\alpha_2 s + \beta_1 + i\beta_2)) \left(\frac{\gamma_1}{\beta_1 + i\beta_2} \right) \left(\frac{\gamma_1}{\beta_1 - i\beta_2} \right) (\alpha_1 s - i\alpha_2 s + \beta_1 - i\beta_2)) \\ &\qquad = ((a_1 s + \gamma_1) + i(a_2 s + \gamma_2))((a_1 s + \gamma_1) - i(a_2 s + \gamma_2)) \\ &\qquad = (a_1 s + \gamma_1)^2 + (a_2 s + \gamma_2)^2, \end{aligned}$$

with $a_1, a_2 \in \mathbb{R}\{s,t\}$. Now, using that $(s)\mathbb{R}\{s,t\} \subset \operatorname{im}\psi_n$ and that $\gamma_1(t) = g_1 \circ \varphi_n(0,t) = g_1 \circ \varphi_n(s,t)$, it is easy to conclude that there exist $h_1, h_2 \in \mathcal{O}(Y_n)$ such that

$$f \equiv (h_1 + g_1)^2 + h_2^2 \mod \mathcal{J}(Y_n)$$

and therefore, $\mathcal{P}(Y_n) = \Sigma_2(Y_n)$. ■

We finish with the surface germs Z_n:

Theorem 4.5 *The surface germs Z_n, $n \geq 3$, have $p = 2$ and $\mathcal{P} \neq \Sigma$.*

Proof. The parametrization

$$\phi_n : (s,t) \mapsto (s, st, \ldots, st^{n-2}, t^{n-1}, t^n),$$

defines the homomorphism of rings

$$\begin{array}{rccc} \phi_n^* : & \mathcal{O}(Z_n) & \to & \mathbb{R}\{s,t\} \\ & f & \mapsto & f \circ \phi_n. \end{array}$$

which is finite, injective and $(s)\ \mathbb{R}\{s,t\} \subset \operatorname{im}\phi_n^*$. For this last fact, note that $s^i t^j = s^{i-1}(st^r)t^{(n-1)q} = \phi_n^*(x_0^{i-1} x_r x_{n-1})$ where $j = (n-1)q + r$, $0 \leq r < n-1$, $i \geq 1$.

We now check that $p[Z_n] = 2$. Let $f \in \Sigma(Z_n)$ and consider $f \circ \phi_n$. Since f is an sos in $\mathcal{O}(Z_n)$ then $f \circ \phi_n$ is an sos in $\mathbb{R}\{s,t\}$ and there exist $\alpha_1, \alpha_2, \in \mathbb{R}\{s,t\}$ and $\beta_1, \beta_2 \in \mathbb{R}\{t\}$ such that

$$f \circ \phi_n = (\alpha_1 s + \beta_1)^2 + (\alpha_2 s + \beta_2)^2.$$

It is clear that $C_n = Z_n \cap \{x_0 = 0\}$ is the planar curve parametrized by $\phi_n(0,t) = (0, \ldots, 0, t^{n-1}, t^n)$. This curve has ideal $\mathcal{J}(C_n) = (x_0, \ldots, x_{n-2}, x_n^{n-1} - x_{n-1}^n)$ and Pythagoras number 2. Thus, for $g = f|_{C_n} \in \Sigma(C_n)$, we find $g_1, g_2 \in \mathbb{R}\{x_{n-1}, x_n\}$ such that

$$g \equiv g_1^2 + g_2^2 \mod \mathcal{J}(C_n).$$

Hence, if $\gamma_i = g_i \circ \phi_n(0,t)$, $i = 1, 2$, we deduce

$$\beta_1^2 + \beta_2^2 = f \circ \phi_n(0,t) = f|_{C_n} \circ \phi_n(0,t) = \gamma_1^2 + \gamma_2^2,$$

and $\dfrac{\gamma_1 + i\gamma_2}{\beta_1 + i\beta_2}, \dfrac{\gamma_1 - i\gamma_2}{\beta_1 - i\beta_2}$ are two units in $\mathbb{C}\{t\}$ whose product is 1. Consequently,

$$\begin{aligned} &(\alpha_1 s + \beta_1)^2 + (\alpha_2 s + \beta_2)^2 \\ &\qquad = (\alpha_1 s + i\alpha_2 s + \beta_1 + i\beta_2)) \left(\frac{\gamma_1 + i\gamma_2}{\beta_1 + i\beta_2}\right) \left(\frac{\gamma_1 - i\gamma_2}{\beta_1 - i\beta_2}\right) (\alpha_1 s - i\alpha_2 s + \beta_1 - i\beta_2)) \\ &\qquad = ((a_1 s + \gamma_1) + i(a_2 s + \gamma_2))((a_1 s + \gamma_1) - i(a_2 s + \gamma_2)) \\ &\qquad = (a_1 s + \gamma_1)^2 + (a_2 s + \gamma_2)^2, \end{aligned}$$

with $a_1, a_2 \in \mathbb{R}\{s,t\}$. Now, using that $(s)\ \mathbb{R}\{s,t\} \subset \operatorname{im}\psi_n$ and that $\gamma_i(t) = g_i \circ \phi_n(0,t) = g_i \circ \phi_n(s,t)$, it is easy to conclude that there exist $h_1, h_2 \in \mathcal{O}(Z_n)$ such that

$$f \equiv (h_1 + g_1)^2 + (h_2 + g_2)^2 \mod \mathcal{J}(Z_n)$$

and therefore, $p[Z_n] = 2$.

Finally, $\mathcal{P}(Z_n) \neq \Sigma(Z_n)$. Let

$$f = \begin{cases} x_{n-1} & \text{if } n \text{ is odd} \\ x_n & \text{if } n \text{ is even.} \end{cases}$$

Clearly, $f \in \mathcal{P}(Z_n)$ is not an sos in $\mathcal{O}(Z_n)$. ■

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Depto. Algebra
F. Ciencias Matemáticas
Universidad Complutense de Madrid
28040 Madrid, Spain
e-mail: josefer@mat.ucm.es