Answer FOUR of the FIVE questions. If more than four questions are attempted, then credit will be given for the best four answers.

*Show all your work and justify your answers.*
1. [20 marks]

Let $G$ be a finite group.

(a) Define a $G$-space, a $G$-subspace and the character of a $G$-space.

(b) Prove that if $A$ is a finite abelian group then every irreducible $A$-space has dimension 1.

(c) Let $V$ be a $G$-space and let $g \in G$. Show that there is a basis for $V$ such that the matrix for $g$ is diagonal with roots of unity on the diagonal. (You may assume Maschke’s theorem, that every $G$-space is a direct sum of irreducible $G$-spaces.)

Let $\chi$ be a character of $G$.

(d) Show that $|\chi(g)| \leq \chi(1)$ for any $g \in G$.

(e) If $g$ is conjugate to $g^{-1}$, show that $\chi(g) \in \mathbb{R}$.

(f) If $g$ is of order 2, show that $\chi(g) \in \mathbb{Z}$ and that $\chi(g) \equiv \chi(1) \pmod{2}$.

(g) If $g$ is of order 3 and $g$ is conjugate to $g^{-1}$, show that $\chi(g) \in \mathbb{Z}$ and that $\chi(g) \equiv \chi(1) \pmod{3}$.

2. [20 marks]

(a) For a finite group $G$, define a homomorphism of $G$-spaces.

(b) State and prove Schur’s Lemma.

Let $V$ be a $G$-space, $\rho_v$ an associated matrix representation with character $\chi_v$, and $f$ a class function on $G$. Define a matrix $P_f$ by

$$P_f = \frac{\dim V}{|G|} \sum_{g \in G} f(g) \rho_v(g).$$

(c) Show that multiplication by $P_f$ is a homomorphism from the $G$-space $V$ to itself.

(d) If $V$ is irreducible, deduce that $P_f = \lambda \text{Id}_V$ for some $\lambda \in \mathbb{C}$. Show that $\lambda = (\chi_v, f)$.

(e) Still assuming $V$ to be irreducible, let $W$ be another irreducible $G$-space. Quoting without proof any results from the course that you need, show that $P_{\chi_W} = \begin{cases} \text{Id}_V & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$

(f) If $V$ is not irreducible, write it as a sum of irreducibles: $V \cong \bigoplus_i V_i$. Show that $P_{\chi_W}$ is projection onto the sum of the $V_i$ with $V_i \cong W$. 

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3. [20 marks]

(a) Define a class function on a finite group $G$ and the inner product $(\phi, \mu)_G$ of two class functions $\phi$ and $\mu$.

(b) Prove that characters of $G$-spaces are class functions.

(c) If $H$ is a subgroup of $G$ and $\theta$ is a class function on $H$, write down the formula for the induced class function $\text{Ind}_H^G \theta$ on $G$.

(d) State and prove the Frobenius Reciprocity Theorem for class functions.

Let $H$ be a subgroup of $G$ and let $V$ be an irreducible $G$-space. Let $U$ be an irreducible $H$-subspace of $\text{Res}_H^G V$.

(e) Use Frobenius reciprocity to show that $V$ is isomorphic to a $G$-subspace of $\text{Ind}_H^G U$ and deduce that $\dim V \leq |G : H| \dim U$.

(f) Deduce that $\dim V \leq |G : H|$ when $H$ is abelian.

(g) Returning to the case when $H$ is not necessarily abelian, let $W$ be an irreducible $H$-space and $X$ an irreducible $G$-subspace of $\text{Ind}_H^G W$. Show that $W$ is isomorphic to an $H$-subspace of $\text{Res}_H^G X$.

4. [20 marks]

(a) Let $X$ be a finite $G$-set and $\mathbb{C}X$ the corresponding permutation $G$-space. State and prove a formula for $\chi_{\mathbb{C}X}(g)$ in terms of the action of $g$ on $X$.

(b) Write down representatives for the conjugacy classes of $S_4$, the symmetric group on 4 objects, and calculate the size of each class. How many irreducible representations does $S_4$ have?

(c) Find the two 1-dimensional characters of $S_4$.

From now on be careful to explain all the properties of characters that you use.

(d) Calculate the character of the natural permutation representation of $S_4$ and use it to produce a 3-dimensional irreducible character of $S_4$.

(e) Use standard properties of characters to complete the character table of $S_4$. 

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5. [20 marks]

Let \( G \) be a finite group and let \( \chi_1, \ldots, \chi_n \) be the irreducible characters of \( G \).

(a) Write down two formulas that express row orthogonality and column orthogonality for the character table of \( G \).

(b) Use one of these to show that \( \sum_{i=1}^{n} \chi_i(1)^2 = |G| \).

(c) Suppose that \( g \in G \) is such that \( g \neq 1 \). Show that if all the values \( \chi_i(g) \) are real, then there is some \( i \) such that \( \chi_i(g) < 0 \).

(d) Prove that, for any character \( \chi \) and any \( g \in G \), \( \chi(g^{-1}) = \overline{\chi(g)} \). You may quote any other results from the course without proof.

(e) If \( g \in G \) is not conjugate to \( g^{-1} \), show that \( \sum_{i=1}^{n} \chi_i(g)^2 = 0 \).

(f) Prove that if \( g \) is not conjugate to \( g^{-1} \), then for some irreducible character, \( \chi_i \), the value \( \chi_i(g) \) is not a real number.