3 hours

THE UNIVERSITY OF MANCHESTER

HYPERBOLIC GEOMETRY

13 January 2014
2.00 - 5.00

Answer ALL FOUR questions in Section A (40 marks in total).
Answer TWO of the THREE questions in Section B (60 marks in total).
Answer ALL THREE questions in Section C (50 marks in total).
If more than TWO questions from Section B are attempted then credit will be given for the TWO best answers.

Electronic calculators are permitted provided that they cannot store text.

Notation: Throughout, $\mathbb{H}$ denotes the upper half-plane, $\partial \mathbb{H}$ denotes the boundary of $\mathbb{H}$, $\mathbb{D}$ denotes the Poincaré disc, and $\partial \mathbb{D}$ denotes the boundary of $\mathbb{D}$. 
SECTION A

Answer **ALL** four questions

A1.

(i) Let $A$ be either a straight line or a circle in $\mathbb{C}$. Prove that points on $A$ satisfy an equation of the form

$$\alpha z\bar{z} + \beta z + \bar{\beta} \bar{z} + \gamma = 0 \quad (*)$$

where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$. 

[6 marks]

(ii) Geodesics in $\mathbb{H}$ correspond to vertical straight lines in $\mathbb{H}$ and semi-circles with real centres in $\mathbb{H}$. These have equations of the form $(*)$ with $\beta \in \mathbb{R}$ (you do not need to prove this).

Consider the points $-5 + 12i$ and $12 + 5i$ in $\mathbb{H}$. Determine an equation of the form $(*)$ (i.e. find values of $\alpha, \beta, \gamma \in \mathbb{R}$) that describes the geodesic through these two points.

[4 marks]

A2.

(i) Let $\sigma : [a, b] \rightarrow \mathbb{H}$ be a path in $\mathbb{H}$. Define the hyperbolic length, $\text{length}_\mathbb{H}(\sigma)$, of $\sigma$.

[2 marks]

(ii) Show that the hyperbolic length of the vertical straight line from $1 + 3i$ to $1 + 12i$ is $\log 4$.

[4 marks]

A3.

(i) Let $S = \{a_1, \ldots, a_k\}$ be a finite set of symbols. Briefly explain how to construct the free group on $k$ generators $F_k$.

(Your answer should include: a description of the elements of $F_k$, a description of the group operation, a description of the group identity, and a brief explanation of how to find the inverse of an element in $F_k$. You do not need to prove that the group operation is well-defined.)

[4 marks]

(ii) Consider $F_2$, the free group on 2 generators $a, b$. Show that there are 4 words of length 1 and 12 words of length 2 in $F_2$.

How many words of length $n$ are there in $F_2$?

[6 marks]
A4.

(i) Consider the regular hyperbolic decagon in Figure 1 below with each internal angle equal to $\pi/9$ and with the sides paired as illustrated (you may assume that such a hyperbolic decagon exists).

Show that there are two elliptic cycles and determine their orders. By using Poincaré's Theorem, show that the side pairing transformations generate a co-compact Fuchsian group $\Gamma$. (You do not need to give a presentation of $\Gamma$ in terms of generators and relations.)

[10 marks]

(ii) Write down the signature $\text{sig}(\Gamma)$ of $\Gamma$. Sketch a picture of the quotient space $\mathbb{H}/\Gamma$.

[4 marks]

Figure 1: See Question A4. Each internal angle is $\pi/9$ and the sides are paired as indicated.
B5. Let $\Gamma$ be a Fuchsian group acting on either $\mathbb{H}$ or $\mathbb{D}$. Recall that an open set $F \subset \mathbb{H}$ (or $\mathbb{D}$) is a fundamental domain if

- $\bigcup_{\gamma \in \Gamma} \gamma(\text{cl}(F)) = \mathbb{H}$ (or $\mathbb{D}$)
- $\gamma_1(F) \cap \gamma_2(F) = \emptyset \text{ if } \gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2$.

Consider the following statements. In each case, state whether the statement is true or false and justify your answer by giving either a proof or a counterexample. You may use any of the results from the course, provided that you state them clearly.

(i) The set

$$\{z \in \mathbb{H} \mid |z| \geq 1\}$$

is a fundamental domain for some Fuchsian group. [2 marks]

(ii) There exists a Fuchsian group with

$$\{z \in \mathbb{H} \mid -2 < \text{Re}(z) < 2\}$$

as a fundamental domain. [4 marks]

(iii) Let $\Gamma$ be a Fuchsian group. Then there exists a unique fundamental domain for $\Gamma$. [4 marks]

(iv) The perpendicular bisector of the arc of geodesic $[z_1, z_2]$ between $z_1, z_2 \in \mathbb{H}$ is given by

$$\{z \in \mathbb{H} \mid d_\mathbb{H}(z, z_1) = d_\mathbb{H}(z, z_2)\}.$$ [6 marks]

(v) The modular group $\text{PSL}(2, \mathbb{Z})$ has a fundamental domain with hyperbolic area $\pi/8$. [4 marks]

(vi) The modular group $\text{PSL}(2, \mathbb{Z})$ is generated by $z \mapsto z + 1, z \mapsto -1/z$. [4 marks]

(vii) Let $D$ be any convex hyperbolic polygon. Then $D$ is a Dirichlet polygon for some Fuchsian group. [6 marks]
B6.

(i) Let $\gamma(z) = (az + b)/(cz + d)$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$ be a Möbius transformation of $\mathbb{H}$. Prove the following two identities:

$$\gamma'(z) = \frac{ad - bc}{(cz + d)^2}, \quad \text{Im}(\gamma(z)) = \frac{ad - bc}{|cz + d|^2} \text{Im}(z).$$

Conclude that $\text{Im}(\gamma(z)) = |\gamma'(z)| \text{Im}(z)$.

[6 marks]

(ii) Let $z, w \in \mathbb{H}$ and let $\gamma \in \text{Möb}(\mathbb{H})$. By using the fact that $d_{\mathbb{H}}(ia, ib) = \log b/a$ when $0 < a < b$, the fact that

$$|\gamma(z) - \gamma(w)| = |z - w| \sqrt{|\gamma'(z)| |\gamma'(w)|}$$

(you do not need to prove this yourself), and the results in (i) above, prove that

$$\cosh d_{\mathbb{H}}(z, w) = 1 + \frac{|z - w|^2}{2 \text{Im}(z) \text{Im}(w)},$$

stating clearly any standard results from the course that you use. (You may assume that if $0 < a < b$ then $d_{\mathbb{H}}(ia, ib) = \log b/a$.)

[10 marks]

(iii) Let $\Delta$ be a hyperbolic right-angled triangle with side lengths $a, b, c$ where $c$ is the length of the side opposite the right-angle.

State and prove the hyperbolic version of Pythagoras' Theorem. (If you reduce the problem to a particular case then you should briefly explain why this reduction is valid.)

[8 marks]

(iv) Consider a hyperbolic quadrilateral where two opposing angles are right-angles (you may assume that such quadrilaterals exist) and with side lengths as indicated in Figure 2.

![Figure 2](https://via.placeholder.com/150)

Figure 2: See Question B6(iv).

Prove that $\cosh a_1 \cosh b_1 = \cosh a_2 \cosh b_2$.

What is the Euclidean analogue of this identity?

[6 marks]
B7.

(i) Let \( \gamma(z) = (az + b)/(cz + d), \) \( a, b, c, d \in \mathbb{R}, \) \( ad - bc > 0 \) be a Möbius transformation of \( \mathbb{H}. \)

In terms of fixed points, what does it mean to say that \( \gamma \) is parabolic?

Suppose that \( c \neq 0. \) Prove that \( \gamma \) is parabolic if and only if \( (d-a)^2 + 4bc = 0. \) [6 marks]

For the remainder of this question, let \( k, \ell \in \mathbb{R} \) and define

\[
\gamma_1(z) = \frac{3z + (k - 3)}{2z + (k - 2)}, \quad \gamma_2(z) = \frac{3z - (2\ell + 6)}{2z - (\ell + 4)}.
\]

(ii) For which values of \( k, \ell \) are \( \gamma_1, \gamma_2 \) Möbius transformations of \( \mathbb{H}? \)

For which values of \( k, \ell \) are \( \gamma_1, \gamma_2 \) parabolic? [8 marks]

(iii) In the case when \( \gamma_1, \gamma_2 \) are Möbius transformations of \( \mathbb{H} \) show, by considering end-points, that \( \gamma_1 \) maps \( s_1 \) to \( s_2 \) and \( \gamma_2 \) maps \( s_4 \) to \( s_3 \), as illustrated in Figure 3.

![Figure 3: See Question B7(iii).](image-url)

(iv) What does it mean for a parabolic cycle to satisfy the parabolic cycle condition?

Determine the values of \( k, \ell \) for which \( \gamma_1, \gamma_2 \) generate a Fuchsian group \( \Gamma. \) Give a presentation of \( \Gamma \) in terms of generators and relations. [12 marks]
SECTION C

Answer ALL three questions

C8. Let $X$ be a metric space and let $\Gamma$ be a group of homeomorphisms that acts on $X$.

(i) Let $x \in X$. Define the stabiliser $\text{Stab}_\Gamma(x)$ of $x$ in $\Gamma$. [2 marks]

(ii) Let $g : X \to X$ be a homeomorphism. Show that $\text{Stab}_\Gamma(g(x)) = g \text{Stab}_\Gamma(x)g^{-1}$. [4 marks]

Now consider the case when $X = \mathbb{H}$ and $x = i \in \mathbb{H}$.

(iii) Show that

$$\text{Stab}_{\text{Möb}(\mathbb{H})}(i) = \left\{ z \mapsto \frac{(\cos \theta)z + \sin \theta}{(-\sin \theta)z + \cos \theta} \mid \theta \in [0, \pi) \right\}.$$ 

Calculate $\text{Stab}_{\text{PSL}(2,\mathbb{Z})}(i)$. [8 marks]

(iv) Find a Möbius transformation $g \in \text{Möb}(\mathbb{H})$ such that $g(i) = 1 + i$. Hence, using (ii), show that

$$\text{Stab}_{\text{Möb}(\mathbb{H})}(1+i) = \left\{ z \mapsto \frac{(\cos \theta - \sin \theta)z + 2\sin \theta}{(-\sin \theta)z + (\sin \theta + \cos \theta)} \mid \theta \in [0, \pi) \right\}.$$ 

and calculate $\text{Stab}_{\text{PSL}(2,\mathbb{Z})}(1+i)$. [8 marks]

C9. Let $X$ be a proper metric space and let $\Gamma$ be a group of homeomorphisms that acts on $X$.

(i) What does it mean to say that $\Gamma$ acts properly discontinuously on $X$? [2 marks]

(ii) Suppose that $\Gamma$ acts properly discontinuously on $X$. Let $x \in X$. Prove that the orbit $\Gamma(x) \subset X$ is a discrete set. Prove that the stabiliser $\text{Stab}_\Gamma(x)$ of $x$ is finite. [8 marks]

Let $X = \partial \mathbb{H}$, the boundary of the upper half-plane.

(iii) Let $\Gamma = \text{PSL}(2,\mathbb{Z})$ be the modular group. Then $\Gamma$ acts on $\partial \mathbb{H}$ by homeomorphisms. Show that $\Gamma$ does not act properly discontinuously on $\partial \mathbb{H}$. [4 marks]

(iv) Let $\Gamma = \{\gamma_n \mid \gamma_n(x) = x + n\}$ be the group of integer translations. Then $\Gamma$ acts on $\partial \mathbb{H}$ by homeomorphisms. Does $\Gamma$ act properly discontinuously on $\partial \mathbb{H}$? [4 marks]

7 of 8

P.T.O.
C10.

(i) Let $\Gamma$ be a Fuchsian group acting on $\mathbb{H}$ and let $z \in \mathbb{H}$. What is meant by a limit point of $\Gamma(z)$? [2 marks]

(ii) Suppose that $\gamma \in \Gamma$ is hyperbolic. Prove that the fixed points of $\gamma$ are in $\Lambda(\Gamma)$. (You may use any standard results from the course provided that you state them clearly.) [6 marks]

(iii) Determine the limit set of the Fuchsian group $\Gamma$ acting on $\mathbb{H}$ where

$$\Gamma = \left\{ \gamma_n \mid \gamma_n(z) = \frac{2^n z}{(1 - 2^n)z + 1} \right\}.$$ [4 marks]