Three Hours

THE UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

23 January 2014
9.45 – 12.45

Answer ALL FIVE questions in Section A (56 marks in all).
Answer TWO of the THREE questions in Section B (24 marks in total).
If more than TWO questions from Section B are attempted,
then credit will be given for the FIRST TWO answers.

A list of axioms and rules of proof is appended to this examination paper

__________________________________

Electronic calculators are not permitted

__________________________________
A1. Let the language $L$ have a binary relation symbol $R$ and a binary function symbol $f$. Which of the following are terms of $L$? You should justify your answers.

(i) $f(x_1, f(x_1, x_1))$
(ii) $f((f(x_1, x_2), x_1))$

Which of the following are formulae of $L$? You should briefly justify your answers.

(iii) $\forall w_1 (R(x_1, x_1) \lor R(x_1, x_1))$
(iv) $\forall x_1 (R(x_1, x_1) \lor \neg R(x_1, x_1))$

Let $M$ be the structure for $L$ with $|M| = \{2, 3, 4, \ldots\}$, $f_M(n, m) = n \times m$, and $R_M = \{ \langle n, m \rangle \in |M|^2 \mid n < m \}$.

Which of the following sentences of $L$ are true in $M$?

(v) $\forall w_1 \exists w_2 R(w_2, w_1)$,
(vi) $\forall w_1 \forall w_2 (\exists w_3 R(f(w_1, w_3), f(w_2, w_3)) \rightarrow R(w_1, w_2))$,
(vii) $\exists w_1 \forall w_2 (R(w_2, w_1) \rightarrow R(f(w_2, w_2), w_1))$.

Find formulae $\theta_1(x_1, x_2)$, $\theta_2(x_1)$, $\theta_3(x_1, x_2)$, $\theta_4(x_1)$ of $L$ such that for $n \in |M|,

$M \models \theta_1(n, m) \iff n = m$,
$M \models \theta_2(n) \iff n = 2$,
$M \models \theta_3(n, m) \iff n^2 \leq m$,
$M \models \theta_4(n) \iff n = 3$.

Let $K$ be the structure for $L$ with $|K| = \mathbb{N} = \{1, 2, 3, \ldots\}$, $f^K(n, m) = n \times m$, $R^K = \{ \langle n, m \rangle \in |K|^2 \mid n < m \}$.

Find a sentence $\phi$ of $L$ such that $K \models \phi$ and $M \not\models \phi$.

A2. Write down a sentence in Prenex Normal Form logically equivalent to

$(\exists w_1 P(w_1) \rightarrow \neg (\forall w_1 P(w_1) \land \exists w_1 Q(w_1)))$

where $P, Q$ are unary relation symbols.

A3. Let $L$ be a relational language and for $\theta \in FL$ let $\theta^*$ be the expression resulting from removing every occurrence of $\neg$ in $\theta$. Outline a proof that for $\theta \in FL$, $\theta^* \in FL$. 
A4. Define what is meant by a formal proof. Give a formal proof of
\[ \exists w_1 (\theta(w_1) \rightarrow \phi) \vdash \forall w_1 \theta(w_1) \rightarrow \phi \]
where \( w_1 \) does not occur in \( \phi \).

A5. State the Completeness Theorem. Using this theorem or otherwise show that
(a) \( \forall w_1 P(g(w_1)) \not\vdash \forall w_1 P(w_1) \)
(b) \( \forall w_1 (P(w_1) \rightarrow \neg P(g(w_1))) \vdash \exists w_1 \neg P(w_1) \)
where \( P \) is a unary relation symbol and \( g \) a unary function symbol.
Does \( P(x_1) \rightarrow \neg P(g(x_1)), P(x_2) \vdash \neg P(g(x_2)) \).
Justify your answer.
B6. (a) Show that
\[ EqL, \forall w_1 \forall w_2 (\theta(w_1, w_2) \rightarrow w_1 = w_2) \models \forall w_1, w_2 (\theta(w_1, w_2) \rightarrow \theta(w_2, w_1)). \]

(b) Give a formal proof of
\[ EqL, \phi(x_1), \neg \phi(x_2) \vdash \neg x_1 = x_2. \]

[12 marks]

B7. Suppose that the language \( L \) has constant symbols \( c, d \) and for \( \theta \) a formula of \( L \) let \( \theta \) be the formula of \( L \) resulting from replacing each occurrence of \( c \) in \( \theta \) by \( d \). \textit{Outline} a proof that if \( \models \theta \) then \( \models \theta \).

Does the result still hold if only some occurrences of \( c \) are replaced by \( d \)? You should justify your answer.

[12 marks]

B8. Describe an infinite set \( \Gamma \) of sentences of the language \( L \) with just equality such that for \( M \) a normal structure for \( L \),
\[ M \models \Gamma \iff |M| \text{ is infinite.} \]

Suppose that \( \Gamma' \subseteq SL \) is another set of sentences with this same property. Show that for every \( \theta \in \Gamma' \) there is a finite subset \( \Delta \) of \( \Gamma \) such that
\[ EqL, \Delta \vdash \theta. \]

Hence show that any such \( \Gamma' \) cannot be finite.

[12 marks]
The Rules of Proof and Axiom for the Predicate Calculus

**And In (AND)**
\[
\Gamma | \theta, \Delta | \phi \\
\Gamma \cup \Delta | \theta \land \phi
\]

**And Out (AO)**
\[
\Gamma | \theta \land \phi \\
\Gamma | \theta \\
\Gamma | \phi
\]

**Or In (ORR)**
\[
\Gamma | \theta \\
\Gamma | \theta \lor \phi \\
\Gamma | \phi \\
\Gamma | \theta
\]

**Disjunction (DIS)**
\[
\Gamma, \theta | \psi, \Delta, \phi | \psi \\
\Gamma \cup \Delta | \theta \lor \phi
\]

**Implies In (IMR)**
\[
\Gamma, \theta | \phi \\
\Gamma | \theta \rightarrow \phi
\]

**Modus Ponens (MP)**
\[
\Gamma | \theta, \Delta | \theta \rightarrow \phi \\
\Gamma \cup \Delta | \phi
\]

**Not In (NIN)**
\[
\Gamma, \theta | \phi, \Delta, \theta | \neg \phi \\
\Gamma \cup \Delta | \neg \theta
\]

**Not Not Out (NNO)**
\[
\Gamma | \neg \neg \theta \\
\Gamma | \theta
\]

**Monotonicity (MON)**
\[
\Gamma | \theta \\
\Gamma \cup \Delta | \theta
\]

**All In (\(\forall I\))**
\[
\Gamma | \theta \\
\Gamma \forall w_j \theta(w_j/x_i)
\]
where \(x_i\) does not occur in any formula in \(\Gamma\) and \(w_j\) does not occur in \(\theta\)

**All Out (\(\forall O\))**
\[
\Gamma | \forall w_j \theta(w_j, \bar{x}) \\
\Gamma | \theta(t(\bar{x}), \bar{x})
\]
for \(t(\bar{x}) \in TL\)

**Exists In (\(\exists I\))**
\[
\Gamma | \theta \\
\Gamma | \exists w_j \theta'
\]
where \(\theta'\) is the result of replacing any number of occurrences of the term \(t(\bar{x})\) in \(\theta\) by \(w_j\) and \(w_j\) does not occur in \(\theta\).

**Exists Out (\(\exists O\))**
\[
\Gamma, \phi | \theta \\
\Gamma, \exists w_j \phi(w_j/x_i) | \theta
\]
where \(x_i\) does not occur in \(\theta\) nor any formula in \(\Gamma\) and \(w_j\) does not occur in \(\phi\).

**REF**
\[
\Gamma | \theta \text{ whenever } \theta \in \Gamma.
\]
The Equality Axioms, $EqL$

**Eq1** \( \forall w_1 w_1 = w_1 \)

**Eq2** \( \forall w_1, w_2 (w_1 = w_2 \rightarrow w_2 = w_1) \)

**Eq3** \( \forall w_1, w_2, w_3 ((w_1 = w_2 \land w_2 = w_3) \rightarrow w_1 = w_3) \)

**Eq4** \( \forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow (R(w_1, w_2, \ldots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right) \)

for \( R \) an \( r \)-ary relation symbol of \( L \).

**Eq5** \( \forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{n+i} \right) \rightarrow f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right) \)

for \( f \) an \( r \)-ary function symbol of \( L \).

**Eq6** \( \forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow t(w_1, w_2, \ldots, w_r) = t(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right) \)

for \( t(x_1, x_2, \ldots, x_r) \in TL \).

**Eq7** \( \forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow (\theta(w_1, w_2, \ldots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right) \)

for \( \theta(x_1, x_2, \ldots, x_r) \in FL \).