3 hours

THE UNIVERSITY OF MANCHESTER

HYPERBOLIC GEOMETRY

13 January 2014
2.00 - 5.00

Answer ALL FOUR questions in Section A (40 marks in total).
Answer TWO of the THREE questions in Section B (60 marks in total).
Answer ALL THREE questions in Section C (50 marks in total).
If more than TWO questions from Section B are attempted then credit will be given for the TWO best answers.

Electronic calculators are permitted provided that they cannot store text.

Notation: Throughout, \( \mathbb{H} \) denotes the upper half-plane, \( \partial \mathbb{H} \) denotes the boundary of \( \mathbb{H} \), \( \mathbb{D} \) denotes the Poincaré disc, and \( \partial \mathbb{D} \) denotes the boundary of \( \mathbb{D} \).
SECTION A

Answer ALL four questions

A1.

(i) Let \( A \) be either a straight line or a circle in \( \mathbb{C} \). Prove that points on \( A \) satisfy an equation of the form

\[
\alpha z \overline{z} + \beta z + \overline{\beta} \overline{z} + \gamma = 0
\]

(\( \ast \))

where \( \alpha, \gamma \in \mathbb{R} \) and \( \beta \in \mathbb{C} \).

[6 marks]

(ii) Geodesics in \( \mathbb{H} \) correspond to vertical straight lines in \( \mathbb{H} \) and semi-circles with real centres in \( \mathbb{H} \). These have equations of the form (\( \ast \)) with \( \beta \in \mathbb{R} \) (you do not need to prove this).

Consider the points \(-5+12i\) and \(12+5i\) in \( \mathbb{H} \). Determine an equation of the form (\( \ast \)) (i.e. find values of \( \alpha, \beta, \gamma \in \mathbb{R} \)) that describes the geodesic through these two points.

[4 marks]

A2.

(i) Let \( \sigma : [a, b] \to \mathbb{H} \) be a path in \( \mathbb{H} \). Define the hyperbolic length, \( \text{length}_\mathbb{H}(\sigma) \), of \( \sigma \).

[2 marks]

(ii) Show that the hyperbolic length of the vertical straight line from \(1+3i\) to \(1+12i\) is \( \log 4 \).

[4 marks]

A3.

(i) Let \( S = \{a_1, \ldots, a_k\} \) be a finite set of symbols. Briefly explain how to construct the free group on \( k \) generators \( \mathcal{F}_k \).

(Your answer should include: a description of the elements of \( \mathcal{F}_k \), a description of the group operation, a description of the group identity, and a brief explanation of how to find the inverse of an element in \( \mathcal{F}_k \). You do not need to prove that the group operation is well-defined.)

[4 marks]

(ii) Consider \( \mathcal{F}_2 \), the free group on 2 generators \( a, b \). Show that there are 4 words of length 1 and 12 words of length 2 in \( \mathcal{F}_2 \).

How many words of length \( n \) are there in \( \mathcal{F}_2 \)?

[6 marks]
(i) Consider the regular hyperbolic decagon in Figure 1 below with each internal angle equal to $\pi/9$ and with the sides paired as illustrated (you may assume that such a hyperbolic decagon exists).

Show that there are two elliptic cycles and determine their orders. By using Poincaré's Theorem, show that the side pairing transformations generate a co-compact Fuchsian group $\Gamma$. (You do not need to give a presentation of $\Gamma$ in terms of generators and relations.)

[10 marks]

(ii) Write down the signature $\text{sig}(\Gamma)$ of $\Gamma$. Sketch a picture of the quotient space $\mathbb{H}/\Gamma$.

[4 marks]

Figure 1: See Question A4. Each internal angle is $\pi/9$ and the sides are paired as indicated.
SECTION B

Answer **TWO** of the three questions

**B5.** Let $\Gamma$ be a Fuchsian group acting on either $\mathbb{H}$ or $\mathbb{D}$. Recall that an open set $F \subset \mathbb{H}$ (or $\mathbb{D}$) is a fundamental domain if

- $\bigcup_{\gamma \in \Gamma} \gamma(cl(F)) = \mathbb{H}$ (or $\mathbb{D}$)
- $\gamma_1(F) \cap \gamma_2(F) = \emptyset$ if $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$.

Consider the following statements. In each case, state whether the statement is true or false and justify your answer by giving either a proof or a counterexample. You may use any of the results from the course, provided that you state them clearly.

(i) The set $\{z \in \mathbb{H} \mid |z| \geq 1\}$

is a fundamental domain for some Fuchsian group.  

[2 marks]

(ii) There exists a Fuchsian group with $\{z \in \mathbb{H} \mid -2 < \text{Re}(z) < 2\}$
as a fundamental domain. 

[4 marks]

(iii) Let $\Gamma$ be a Fuchsian group. Then there exists a unique fundamental domain for $\Gamma$.  

[4 marks]

(iv) The perpendicular bisector of the arc of geodesic $[z_1, z_2]$ between $z_1, z_2 \in \mathbb{H}$ is given by

$\{z \in \mathbb{H} \mid d_\mathbb{H}(z, z_1) = d_\mathbb{H}(z, z_2)\}$. 

[6 marks]

(v) The modular group $\text{PSL}(2, \mathbb{Z})$ has a fundamental domain with hyperbolic area $\pi/8$. 

[4 marks]

(vi) The modular group $\text{PSL}(2, \mathbb{Z})$ is generated by $z \mapsto z + 1, z \mapsto -1/z$. 

[4 marks]

(vii) Let $D$ be any convex hyperbolic polygon. Then $D$ is a Dirichlet polygon for some Fuchsian group. 

[6 marks]

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P.T.O.
B6.

(i) Let \( \gamma(z) = (az + b)/(cz + d) \) where \( a, b, c, d \in \mathbb{R} \) and \( ad - bc > 0 \) be a Möbius transformation of \( \mathbb{H} \). Prove the following two identities:

\[
\gamma'(z) = \frac{ad - bc}{(cz + d)^2}, \quad \text{Im}(\gamma(z)) = \frac{ad - bc}{|cz + d|^2} \text{Im}(z).
\]

Conclude that \( \text{Im}(\gamma(z)) = |\gamma'(z)| \text{Im}(z) \).

[6 marks]

(ii) Let \( z, w \in \mathbb{H} \) and let \( \gamma \in \text{Möb}(\mathbb{H}) \). By using the fact that \( d_{\mathbb{H}}(ia, ib) = \log b/a \) when \( 0 < a < b \), the fact that

\[
|\gamma(z) - \gamma(w)| = |z - w|/(|\gamma'(z)||\gamma'(w)|)
\]

(you do not need to prove this yourself), and the results in (i) above, prove that

\[
cosh d_{\mathbb{H}}(z, w) = 1 + \frac{|z - w|^2}{2 \text{Im}(z) \text{Im}(w)},
\]

stating clearly any standard results from the course that you use. (You may assume that if \( 0 < a < b \) then \( d_{\mathbb{H}}(ia, ib) = \log b/a \).

[10 marks]

(iii) Let \( \Delta \) be a hyperbolic right-angled triangle with side lengths \( a, b, c \) where \( c \) is the length of the side opposite the right-angle.

State and prove the hyperbolic version of Pythagoras' Theorem. (If you reduce the problem to a particular case then you should briefly explain why this reduction is valid.)

[8 marks]

(iv) Consider a hyperbolic quadrilateral where two opposing angles are right-angles (you may assume that such quadrilaterals exist) and with side lengths as indicated in Figure 2.

![Figure 2: See Question B6(iv).](image)

Prove that \( \cosh a_1 \cosh b_1 = \cosh a_2 \cosh b_2 \).

What is the Euclidean analogue of this identity?

[6 marks]
B7.

(i) Let \( \gamma(z) = (az + b)/(cz + d) \), \( a, b, c, d \in \mathbb{R} \), \( ad - bc > 0 \) be a Möbius transformation of \( \mathbb{H} \).

In terms of fixed points, what does it mean to say that \( \gamma \) is parabolic?

Suppose that \( c \neq 0 \). Prove that \( \gamma \) is parabolic if and only if \( (d - a)^2 + 4bc = 0 \). \[6 \text{ marks}\]

For the remainder of this question, let \( k, \ell \in \mathbb{R} \) and define

\[
\gamma_1(z) = \frac{3z + (k - 3)}{2z + (k - 2)}, \quad \gamma_2(z) = \frac{3z - (2\ell + 6)}{2z - (\ell + 4)}.
\]

(ii) For which values of \( k, \ell \) are \( \gamma_1, \gamma_2 \) Möbius transformations of \( \mathbb{H} \)?

For which values of \( k, \ell \) are \( \gamma_1, \gamma_2 \) parabolic? \[8 \text{ marks}\]

(iii) In the case when \( \gamma_1, \gamma_2 \) are Möbius transformations of \( \mathbb{H} \) show, by considering end-points, that \( \gamma_1 \) maps \( s_1 \) to \( s_2 \) and \( \gamma_2 \) maps \( s_4 \) to \( s_3 \), as illustrated in Figure 3.

![Figure 3: See Question B7(iii).]

(iv) What does it mean for a parabolic cycle to satisfy the parabolic cycle condition?

Determine the values of \( k, \ell \) for which \( \gamma_1, \gamma_2 \) generate a Fuchsian group \( \Gamma \). Give a presentation of \( \Gamma \) in terms of generators and relations. \[12 \text{ marks}\]
SECTION C

C8. Let $X$ be a metric space and let $\Gamma$ be a group of homeomorphisms that acts on $X$.

(i) Let $x \in X$. Define the stabiliser $\text{Stab}_\Gamma(x)$ of $x$ in $\Gamma$. [2 marks]

(ii) Let $g : X \to X$ be a homeomorphism. Show that $\text{Stab}_\Gamma(g(x)) = g \text{Stab}_\Gamma(x)g^{-1}$. [4 marks]

Now consider the case when $X = \mathbb{H}$ and $x = i \in \mathbb{H}$.

(iii) Show that

$$\text{Stab}_{\text{Mob}}(i) = \left\{ z \mapsto \frac{(\cos \theta)z + \sin \theta}{(-\sin \theta)z + \cos \theta} \mid \theta \in [0, \pi) \right\}.$$  

Calculate $\text{Stab}_{\text{PSL}(2, \mathbb{Z})}(i)$. [8 marks]

(iv) Find a Möbius transformation $g \in \text{Mob}(\mathbb{H})$ such that $g(i) = 1 + i$. Hence, using (ii), show that

$$\text{Stab}_{\text{Mob}}(1 + i) = \left\{ z \mapsto \frac{(\cos \theta - \sin \theta)z + 2\sin \theta}{(-\sin \theta)z + (\sin \theta + \cos \theta)} \mid \theta \in [0, \pi) \right\}.$$ 

and calculate $\text{Stab}_{\text{PSL}(2, \mathbb{Z})}(1 + i)$. [8 marks]

C9. Let $X$ be a proper metric space and let $\Gamma$ be a group of homeomorphisms that acts on $X$.

(i) What does it mean to say that $\Gamma$ acts properly discontinuously on $X$? [2 marks]

(ii) Suppose that $\Gamma$ acts properly discontinuously on $X$. Let $x \in X$. Prove that the orbit $\Gamma(x) \subset X$ is a discrete set. Prove that the stabiliser $\text{Stab}_\Gamma(x)$ of $x$ is finite. [8 marks]

Let $X = \partial \mathbb{H}$, the boundary of the upper half-plane.

(iii) Let $\Gamma = \text{PSL}(2, \mathbb{Z})$ be the modular group. Then $\Gamma$ acts on $\partial \mathbb{H}$ by homeomorphisms. Show that $\Gamma$ does not act properly discontinuously on $\partial \mathbb{H}$. [4 marks]

(iv) Let $\Gamma = \{ \gamma_n \mid \gamma_n(x) = x + n \}$ be the group of integer translations. Then $\Gamma$ acts on $\partial \mathbb{H}$ by homeomorphisms. Does $\Gamma$ act properly discontinuously on $\partial \mathbb{H}$? [4 marks]
C10.

(i) Let $\Gamma$ be a Fuchsian group acting on $\mathbb{H}$ and let $z \in \mathbb{H}$. What is meant by a limit point of $\Gamma(z)$?

[2 marks]

(ii) Suppose that $\gamma \in \Gamma$ is hyperbolic. Prove that the fixed points of $\gamma$ are in $\Lambda(\Gamma)$. (You may use any standard results from the course provided that you state them clearly.)

[6 marks]

(iii) Determine the limit set of the Fuchsian group $\Gamma$ acting on $\mathbb{H}$ where

$$\Gamma = \left\{ \gamma_n \mid \gamma_n(z) = \frac{2^n z}{(1 - 2^n)z + 1} \right\}.$$ 

[4 marks]

END OF EXAMINATION PAPER

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