THE UNIVERSITY OF MANCHESTER

DIFFERENTIABLE MANIFOLDS

17 January 2014
14:00 – 17:00

Answer **ALL FOUR** questions in Section A (40 marks in total).
Answer **TWO** of the **THREE** questions in Section B (60 marks in total).
Answer **ALL TWO** questions in Section C (40 marks in total).
If more than **TWO** questions in Section B are attempted, the credit will be given for the best **TWO** answers.

Electronic calculators are **not** allowed.

Throughout the paper, where the index notation is used, the **Einstein summation convention** over repeated indices is applied if it is not explicitly stated otherwise.
A1.

(a) For a set $X$, explain what is meant by a chart (or local coordinate system) on $X$ and what is meant by an atlas on $X$.

(b) Define the notion of a change of coordinates between two charts and explain what is meant by a smooth atlas.

(c) Consider the real projective line $\mathbb{R}P^1$ as the set of equivalence classes $(x^1 : x^2)$ of non-zero vectors $(x^1, x^2) \in \mathbb{R}^2 \setminus \{0\}$ with respect to the equivalence relation $(x^1 : x^2) = (\lambda x^1 : \lambda x^2)$ for arbitrary $\lambda \neq 0$. Let $U_i = \{(x^1 : x^2) \mid x^i \neq 0\} \subset \mathbb{R}P^1$, $i = 1, 2$. Define maps $\varphi_i : \mathbb{R} \to U_i$ by $\varphi_1(u) = (1, u)$ and $\varphi_2(v) = (v, 1)$. Each $\varphi_i$ is a chart and the collection $(\varphi_i)_{i=1,2}$ is an atlas for $\mathbb{R}P^1$. [You do not have to show this.] Calculate the change of coordinate $\varphi_{12} = \varphi_1^{-1} \circ \varphi_2$.

A2.

(a) Give the definition of a tangent vector $u$ at a point $x \in M$ in terms of the transformation law for its components $u^i$.

(b) Define the tangent space $T_x M$. Define the vector operations (addition and multiplication by scalars) for the elements of $T_x M$. [You do not have to show that the operations are well-defined.]

(c) For a smooth curve $x = x(t)$ in $M$ define its velocity $\dot{x}$ and show that for all $t$ the velocity $\dot{x}$ is an element of the tangent space $T_{x(t)} M$.

A3.

(a) Explain what is meant by the commutator of vector fields on a manifold. [You do not have to show that the space of vector fields is closed under commutator.] Give a coordinate expression for the commutator of vector fields $u = u^i e_i$ and $w = w^j e_j$. (Here $(e_i)$ is the basis of tangent vectors associated with a local coordinate system, so that $e_i = \frac{\partial}{\partial x^i}$.)

(b) Consider the vector fields on $\mathbb{R}^3$

$$P_1 = \frac{\partial}{\partial x}, \quad P_2 = \frac{\partial}{\partial y}, \quad L = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$  

Calculate their pairwise commutators, that is: $[P_1, P_2]$ and $[L, P_i]$, $i = 1, 2$, expressing the answer in terms of $P_i$ and $L$ themselves.
A4.

(a) Give the axioms defining the \textit{exterior differential}

\[ d: \Omega^p(M) \to \Omega^{p+1}(M), \]

for all \( p = 0, 1, 2, \ldots n \), on a manifold \( M^n \).

(b) Deduce from the axioms the coordinate expression for the differential \( d\sigma \) of a 1-form \( \sigma \) on a 2-dimensional manifold, where

\[ \sigma = \sigma^1 dx^2 - \sigma^2 dx^1 \]

(\( \sigma^1, \sigma^2 \) are given functions).

(c) Explain what is meant by saying that a form \( \omega \) on a manifold \( M \) is \textit{closed} and what is meant by saying that it is \textit{exact}. Show that the 1-form \( \omega = d\theta \) on \( \mathbb{R}^2 \setminus \{0\} \), where \( \theta \) is the polar angle, is closed but not exact.
SECTION B

Answer TWO of the THREE questions

B5.

(a) Explain what is meant by a closed submanifold of dimension \( k \) of a smooth manifold \( M^n \).
Consider a subset \( \Gamma \subset \mathbb{R}^n \) specified by an equation \( F(x^1, \ldots, x^n) = 0 \) such that \( dF(x) \neq 0 \) for all \( x = (x^1, \ldots, x^n) \in \Gamma \). Show that for every \( x \in \Gamma \) there exist local coordinates \( y^1, \ldots, y^n \) defined in some open neighborhood \( W \) of \( x \) such that \( \Gamma \cap W \) is specified by the equation \( y^n = 0 \).

(b) Explain what is meant by saying that a system of equations

\[
f^\mu(x) = 0
\]

in \( \mathbb{R}^n \), where \( \mu = 1, \ldots, k \), has constant rank.
State without proof the theorem about the solution set of a system of equations of constant rank.
Explain the practical method of showing that a system (1) has constant rank using the auxiliary linear system

\[
\frac{\partial f^\mu}{\partial x^i}(x_0) v^i = 0,
\]

where \( x_0 \) satisfies the original system \( f^\mu(x) = 0 \).

(c) Consider the subset \( S \subset \mathbb{R}^3 \times \mathbb{R}^3 \) consisting of all pairs \((a, b)\) of vectors such that \( |a| = |b| = 1 \) and \( a \cdot b = 0 \) (we use the standard scalar product on \( \mathbb{R}^3 \)). Show that the set \( S \) has a natural structure of a manifold of dimension 3. Hint: to analyze the auxiliary linear system, choose a unit vector \( n \) orthogonal to \( a \) and \( b \) and expand the velocities of curves in \( S \) over the orthonormal basis \( a, b, n \).

[30 marks]

B6.

(a) Explain what is meant by a derivation over an algebra homomorphism \( \alpha: A_1 \to A_2 \), where \( A_1 \) and \( A_2 \) are commutative associative algebras.
State Hadamard’s lemma in \( \mathbb{R}^n \) (you do not have to give a proof) and deduce from it that, for arbitrary \( x_0 \in \mathbb{R}^n \), all derivations \( D: C^\infty(\mathbb{R}^n) \to \mathbb{R} \) over the evaluation homomorphism \( \text{ev}_{x_0}: C^\infty(\mathbb{R}^n) \to \mathbb{R} \) have the form \( D = \partial_v \) for some \( v \in \mathbb{R}^n \).

(b) Show that the commutator of vector fields satisfies the Jacobi identity.

(c) Consider on the unit circle \( S^1 \) with the standard polar coordinate \( \theta \) the complex-valued vector fields

\[
L_n = e^{-in\theta} \frac{d}{d\theta},
\]

where \( n \in \mathbb{Z} \). Find the commutator \([L_n, L_m]\) for arbitrary \( n \) and \( m \) and show that it can be expressed as a linear combination with constant coefficients of the vector fields \( L_k, k \in \mathbb{Z} \).
B7.

(a) Explain what is meant by a partition of unity on a manifold $M$ and when a partition of unity $(f_\alpha)$ is said to be subordinate to an open cover $\mathcal{U} = (U_\alpha)$. For a compact manifold $M$, show the existence of a finite partition of unity subordinate to an arbitrary open cover. [You do not need to prove the existence of bump functions.]

(b) Give the definition of the integral of an $n$-form $\omega$ over a compact oriented manifold $M^n$. Prove that the integral $\int_M \omega$ does not depend on choices of an atlas (with a given orientation) and a partition of unity. [You may assume without proof that the integral over a single coordinate domain does not depend on a choice of coordinates as long as orientation is fixed.]

(c) Calculate the integral of the 1-form on $\mathbb{R}^2$

$$\omega = r^\alpha(xdy - ydx)$$

over a circle $x^2 + y^2 = R^2$. Here $x$ and $y$ are standard coordinates, $r^2 = x^2 + y^2$, and $\alpha$ is a parameter. For which values of $\alpha$ the integral does not depend on the radius $R$?

[30 marks]
SECTION C

Answer ALL TWO questions

C8.

(a) Explain what is meant by an an embedding $F: M \to N$.

(b) State, without proof, the corollary from Sard’s Lemma.

(c) Assuming without proof the existence of an embedding of any manifold $M^n$ of dimension $n$ into some $\mathbb{R}^N$, where $N > n$ is sufficiently large, deduce that it is possible, by successive steps, to reduce the dimension of the ambient space and obtain an embedding $M^n \to \mathbb{R}^{2n+1}$.

[20 marks]

C9.

(a) Define what is meant by the de Rham cohomology group $H^k(M^n)$. Define the addition and the multiplication by scalars for the de Rham cohomology classes. [You do not have to show that the operations are well-defined.]

(b) Consider a smooth map $F: M \to N$. Give the definition of the pull-back (or induced map) $F^*: H^k(N) \to H^k(M)$ and show that it is well-defined.

(c) Using the fact that homotopic maps induce the same map on cohomology (you do not have to prove that), deduce that manifolds $M$ and $N$ that are homotopy equivalent have isomorphic de Rham cohomology groups: $H^k(M) \cong H^k(N)$, for all $k$.

(d) Calculate the de Rham cohomology of the manifold $\mathbb{R}^n \setminus \{0\}$. (You may use without proof information about the cohomology of the sphere $S^n$.) Show that $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^m \setminus \{0\}$ for $n \neq m$ cannot be homotopy equivalent.

[20 marks]