Two Hours

Mathematical formula books and statistical tables are to be provided

THE UNIVERSITY OF MANCHESTER

STATISTICAL MODELLING IN FINANCE

17 January 2014
09:45 – 11:45

Answer BOTH questions in Section A.
Answer TWO of the THREE questions in Section B.

If more than TWO questions from section B are attempted
then credit will be given to the best TWO answers.

Electronic calculators may be used, provided that they cannot store text.
A1.

(a) Let \( P_t > 0 \) be the price of an asset at time \( t, t \in \mathbb{N} = \{1, 2, \ldots\} \). Define the (net) return \( R_t \) at time \( t \) and express the log return \( r_t = \log P_t - \log P_{t-1} \) in terms of \( R_t \).

[2+2 marks]

(b) Consider a portfolio of \( n \) assets with weights \( w_i, i = 1, \ldots, n \) at time \( t-1 \). Express the portfolio (net) return \( R_t \) at time \( t \) in terms of the asset (net) returns \( R_{it}, i = 1, \ldots, n \). What transform do you apply to the (net) returns to turn them into log returns? Write down the portfolio log return \( r_t \) at time \( t \) in terms of the asset log returns \( r_{it}, i = 1, \ldots, n \).

[4+2+4 marks]

(c) What is the random walk hypothesis for financial markets? What is the implication on asset prices?

[2+2 marks]

(d) Under the random walk hypothesis with normal increments and no drift, is tomorrow’s price expected to go up or is it expected to go down? Give a reason for your answer.

[2 marks]

[20 marks in total for this question.]
A2. Suppose that a risk-free asset returns $\mu_0 = 4\%$, and that two risky assets have mean returns $\mu_1 = 6\%$ and $\mu_2 = 8\%$ with standard deviations $\sigma_1 = 0.05$ and $\sigma_2 = 0.1$, respectively. The correlation between the two stock returns is $\rho = -0.2$.

(a) Show that the tangency portfolio is given by $w_T = (\gamma_T, 1 - \gamma_T)'$ with

$$
\gamma_T = \frac{(\mu_1 - \mu_0)\sigma_2^2 - (\mu_2 - \mu_0)\rho\sigma_1\sigma_2}{(\mu_1 - \mu_0)\sigma_2^2 + (\mu_2 - \mu_0)\sigma_1^2 - (\mu_1 - \mu_0 + \mu_2 - \mu_0)\rho\sigma_1\sigma_2}.
$$

[4 marks]

(b) Calculate the mean return $\mu_T$ and standard deviation $\sigma_T$ of the tangency portfolio. Do the answers depend on $\rho$?

[2+2+2 marks]

(c) Sketch the feasible region and efficient frontier of all portfolios consisting of the three assets allowing short selling. Indicate the risk-free asset and the tangency portfolio on the graph.

[4 marks]

(d) Find a portfolio on the efficient frontier with mean return $\mu$ and standard deviation $\sigma$ satisfying $\mu = 0.03 + 1.96\sigma$. Assuming a normal distribution for the return on this portfolio, give a 95% lower bound for the return.

[6 marks]

[20 marks in total for this question.]
SECTION B

Answer TWO of the three questions

B1.

(a) Define skewness and (excess) kurtosis. Find the skewness and kurtosis of $Y \sim \text{Exp}(a)$ with probability density function

$$f_Y(y) = ae^{-ay}, \quad y > 0.$$  

[2+2+2+2 marks]

(b) Let $X = e^Y$, where $Y$ is as given in (a). What is the distribution of $X$ called? Find the probability density function of $X$.

[2+2 marks]

(c) Show that the distribution of $X$ has Pareto (right) tail with tail index $a$.

[4 marks]

(d) Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the distribution of $X$ given in (c). Find the maximum likelihood estimator $\hat{a}$ of $a$.

[4 marks]

[20 marks in total for this question.]
B2.

(a) In the Capital Asset Pricing Model (CAPM)

\[ R_i = \mu_0 + \beta_i (R_M - \mu_0) + \varepsilon_i, \ i = 1, 2, \ldots, n, \]

what are the terms \( R_i, R_M \) and \( \varepsilon_i \)? Derive the mean and variance of \( R_i \) stating any assumptions required.

[3+2+2+1 marks]

(b) What role does \( \beta_i \) have in the Security Market Line and what does \( \beta_i^2 \) represent? Give a formula for \( \beta_i \) in terms of the covariance between \( R_i \) and \( R_M \).

[2+2+2 marks]

(c) The following output from R was obtained. The data used were daily closing prices of General Electric (GE) and the S&P500 index from 1-Nov-93 to 31-Oct-94 (250 trading days), and annual percentage Treasury-bill rates.

\[
> \text{data <- read.csv("capm.csv", header=T)} \\
> \text{data <- data[1:250,]} \\
> \text{attach(data)} \\
> \text{fit <- lm(y ~ x)} \\
> \text{summary(fit)}
\]

Coefficients:

| Estimate   | Std. Error | t value | Pr(>|t|) |
|------------|------------|---------|----------|
| (Intercept)| 0.0001444  | 0.000607 | 0.238    | 0.812    |
| x          | 1.0207671  | 0.100163 | 10.197   | <2e-16   |

(i) What model is fitted in the above output? [2 marks]

(ii) Is there significant evidence against the CAPM? Test appropriate hypothesis at the 5% significance level. [2 marks]

(iii) Is there significant evidence that \( \beta \neq 1\)? Test appropriate hypothesis at the 5% significance level and interpret the result. [2 marks]

[20 marks in total for this question.]
B3.

(a) What is Value at Risk (VaR) and what is Expected Shortfall (ES)? What is the connection between them?

[2+2+2 marks]

(b) Write down VaR and ES when the loss distribution is $t_\nu(\mu, \lambda)$, where $\mu$ is location and $\lambda$ is scale parameter. You may use

$$\int_c^{\infty} x f_\nu(x) \, dx = f_\nu(c) \frac{\nu + c^2}{\nu - 1}, \quad c > 0,$$

where $f_\nu$ is the pdf of $t_\nu$.

[2+4 marks]

(c) Give a formula for the semi-parametric estimation of VaR when the loss on an asset has a distribution with a Pareto (right) tail. Explain what the terms are.

[4 marks]

(d) Briefly discuss estimation of VaR for a portfolio of assets.

[4 marks]

[20 marks in total for this question.]