Two Hours

THE UNIVERSITY OF MANCHESTER

ESSENTIAL DIFFERENTIAL EQUATIONS

23 January 2014
2.00 – 4.00

Answer ALL SIX questions in Section A (30 marks in total). Answer TWO of the THREE questions in Section B (20 marks in total). If more than TWO questions from Section B are attempted then credit will be given for the best TWO answers.

Electronic calculators are not permitted.

General Instruction. You may use the Poincaré–Friedrichs inequality in answering any question without giving a proof. The shorthand notation \(|v|\) is used to represent the \(L_2(\Omega)\) norm of a function \(v : \Omega \rightarrow \mathbb{R}\).
SECTION A

Answer ALL questions

Consider the following boundary value problem: find $u$ such that

$$-u''(x) + u'(x) = 1, \quad x \in (0,1) \hspace{1cm} (D)$$

subject to the boundary conditions: $u(0) = 0$, $u'(1) = 0$.

A1. Define the Hilbert space $H^1(0,1)$ by identifying the natural inner product. Show that if $u$ satisfies $(D)$ then it satisfies the variational problem: find $u \in X := \{v \in H^1(0,1); u(0) = 0\}$ such that

$$\int_0^1 u'v' \, dx + \int_0^1 u'v \, dx = \int_0^1 v \, dx \hspace{1cm} (*)$$

for all functions $v \in X$. Explain what is meant by the terms essential boundary condition and natural boundary condition.

[6 marks]

A2. Given square integrable functions $f$ and $g$ defined on $(0,1)$, show that

$$\int_0^1 f g \, dx \leq \left\{ \int_0^1 f^2 \, dx \right\}^{1/2} \left\{ \int_0^1 g^2 \, dx \right\}^{1/2}.$$  

Use this inequality to explain why all three integrals in $(\ast)$ are well defined.

[6 marks]

A3. Consider solving the problem $(\ast)$ using Galerkin’s method. Take a uniform subdivision of the interval $(0,1)$ into two subintervals of length $h = 1/2$, together with a conforming linear finite element approximation space

$$V_2 = \text{span}\{N_1, N_2\},$$

so that the basis functions $N_j$, $j = 1,2$ satisfy the standard interpolation conditions. Plot a graph of the functions $N_1(x)$ and $N_2(x)$. Compute the entries of the $2 \times 2$ Galerkin matrix $A$. (Hint: You should find that $A_{12} \neq A_{21}$.)

[7 marks]

A4. Let $A$ be an $n \times n$ matrix. Define the operator $\exp(A)$. Assuming that $A$ can be diagonalized, (that is, expressed in the form $AU = UA$, where $U$ is an $n \times n$ matrix and $A$ is a diagonal matrix of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A$) show that $\exp(A)$ may be computed without using a series expansion.

[5 marks]
Consider the following PDE problem: given the initial condition \( u_0(x) \), find \( u(x,t) \) satisfying

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in} \ (0,1) \times (0,T],
\]

\[
u(0,t) = 0, \quad u(1,t) = 0, \quad \text{for} \ t > 0,
\]

\[
u(x,0) = u_0(x), \quad \text{for} \ x \in [0,1].
\]

**A5.** Let \( I = [0,T] \) represent the time interval, and suppose that we take a Galerkin approximation space \( X_h \subset X = \{v|v \in H^1(0,1); v(0) = 0, v(1) = 0\} \). Write down the semi-discrete variational problem associated with the PDE problem above; for fixed \( t \in I \), you should define \( u_h(x,t) \in X_h \) to be the solution of the spatially discretized weak formulation.

[2 marks]

**A6.** Following on from **A5**, let \( I \) be divided into time steps \( k_j = t_j - t_{j-1} \) so that \( 0 = t_0 < t_1 < t_2 < \ldots < t_n = T \), and let \( u_h^j \in X_h \) be the implicit approximation associated with the backward difference approximation \( (u_h^j - u_h^{j-1})/k_j \) of the time derivative of \( u_h(x,t_j) \). Write down the resulting fully discrete scheme, and establish the following stability result:

\[
\|u_h^j\| \leq \|u_h^{j-1}\| \leq \ldots \leq \|u_h^0\|.
\]

[4 marks]
SECTION B

Answer **TWO** of the three questions

**B7.** This question uses the notation in question **A1.**

Show that

\[ \int_0^1 v'v \, dx = \frac{1}{2} v(1)^2 \geq 0 \]

for all functions \( v \in X. \)

[3 marks]

State the Lax–Milgram lemma and show that **ALL** conditions of the lemma are satisfied for the variational problem (\( \star \)) with

\[ \|v\|_X := \left[ \int_0^1 v^2 \, dx + \int_0^1 (v')^2 \, dx \right]^{1/2}. \]

[7 marks]
B8. Consider the standard variational problem: find $u \in X := \{v| v \in H^1(0,1); v(0) = 0, v(1) = 0\}$ such that

$$(u', v') = \ell(v), \quad \forall v \in X,$$

where $\ell$ is some given linear functional over the space $X$.

Suppose that we have a finite element approximation space $X_h \subset X$. Show Galerkin orthogonality and establish the best approximation of the Galerkin solution $u_h \in X_h$, that is,

$$\|u - u_h\|_E \leq \|u - v_h\|_E, \quad \forall v_h \in X_h,$$

where $\|v\|_E = \left\{ \int_0^1 (v')^2 \right\}^{1/2}$ is the usual energy norm for functions $v \in X$. [5 marks]

Next, let $z$ be the solution of the corresponding dual problem: find $z$ such that

$$-z''(x) = u(x) - u_h(x), \quad x \in (0,1) \quad (D^*)$$

subject to the boundary conditions: $z(0) = 0, z(1) = 0$. Show that

$$\|u - u_h\|^2 = (u' - u'_h, z' - z'_h)$$

for any $z_h \in X_h$.

Finally, let $h$ be a representative grid parameter and assume that there exists a linear interpolant $z_h^* \in X_h$ satisfying the following estimate:

$$\|z - z_h^*\|_E \leq h\|z''\|.$$

Hence or otherwise, establish the $L_2$ error estimate

$$\|u - u_h\| \leq h\|u - u_h\|_E.$$ [5 marks]
B9. This question uses the notation in question A4.

Show that the vector \( u(t) = \exp(-tA)u(0) \) satisfies the initial value problem: given an initial condition \( u(0) = v \), find \( u(t) \) such that

\[
\frac{du}{dt} + Au = 0 \quad \text{for} \quad t > 0.
\]

Use this problem to explain the meaning of the following statements: (i) the solution operator is \( E(t) = \exp(-tA) \), and (ii) \( E(t) \) has the semigroup property. [5 marks]

Next, consider the generic initial value problem: given a matrix \( A \in \mathbb{R}^{n \times n} \), initial data \( u(0) \in \mathbb{R}^n \) and a forcing function \( f(t) \in \mathbb{R}^n \), we seek \( u(t) \in \mathbb{R}^n \) such that

\[
\frac{du}{dt} + Au(t) = f(t), \quad t \in (0, T].
\]

Show that

\[
\frac{d}{dt} \|u(t)\| \leq A\|u(t)\| + \|f(t)\|,
\]

where \( A = \|A\| \) with \( \| \cdot \| \) representing any norm in \( \mathbb{R}^n \).

The integral form of Gronwall's inequality is that if \( \varphi \) is a nonegative continuous function such that

\[
\varphi(t) \leq a + b \int_0^t \varphi(s) \, ds, \quad \text{for} \quad t > 0
\]

with \( a \) and \( b \) nonnegative constants, then

\[
\varphi(t) \leq ae^{bt}, \quad \text{for} \quad t > 0.
\]

Use this form of Gronwall's inequality to establish the stability estimate

\[
\|u(t)\| \leq e^{At} \left( \|u(0)\| + \int_0^T \|f(s)\| \, ds \right).
\]

[5 marks]