UNIVERSITY OF MANCHESTER

VISCOUS FLUID FLOW

January 14, 2014
14:00 – 16:00

Answer **ALL FIVE** questions (90 marks in all)

Electronic calculators may be used, provided that they cannot store text.
1. The two-dimensional velocity field \( u = u_1 e_1 + u_2 e_2 \) of a viscous Newtonian fluid of density \( \rho \) and dynamic viscosity \( \mu \) that occupies the region \( x_2 \geq 0 \) is given by

\[
\begin{align*}
  u_1 &= x_1^2 x_2 + x_2^2 + A, \\
  u_2 &= B x_1^2 + C x_1 x_2^2,
\end{align*}
\]

where \( A, B \) and \( C \) are constants.

(a) Determine the constants \( A, B \) and \( C \) from the facts that the fluid is (i) in contact with a solid stationary boundary at \( x_2 = 0 \), and (ii) incompressible.

(b) The pressure in the fluid is given by \( p = x_1 + x_2 \). What is the traction that the fluid exerts onto the solid boundary at \( x_2 = 0 \)?

(c) Determine the nonzero component of the vorticity vector \( \omega \) and determine if fluid particles on the solid boundary rotate in the clockwise or anticlockwise direction.

[10 marks]

2. Incompressible viscous fluid of kinematic viscosity \( \nu \) lies at rest in the region \( 0 < y < \infty \). At \( t = 0 \) the rigid boundary at \( y = 0 \) is suddenly set into motion in the \( x \)-direction with constant speed \( U \). You may assume that a parallel flow of the form \( u = u(y,t) e_x \) develops and is governed by the equation

\[
\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.
\]

(a) State the boundary and initial conditions for \( u(y,t) \).

(b) Use linearity and dimensionality arguments to show that

\[
u \frac{\partial^2 u}{\partial y^2} = U f(\eta), \quad \text{where} \quad \eta = \frac{y}{\sqrt{\nu t}}.
\]

(c) Show that \( f(\eta) \) is governed by the ODE

\[
f'' + \frac{1}{2} \eta f' = 0
\]

and transform the boundary and initial conditions into boundary conditions for \( f(\eta) \).

(d) Hence show that

\[
u \frac{\partial^2 u}{\partial y^2} = U \int_\eta^\infty \exp(-\xi^2/4) d\xi.
\]

[**Hint:** You may use the result \( \int_0^\infty \exp(-\xi^2/4) d\xi = \sqrt{\pi} \).]

[15 marks]
3.

(a) Show that in the absence of body forces, the steady, parallel flow of a viscous, incompressible fluid, with velocity $u \mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector in the $x$-direction, is governed by

$$\tilde{\nabla}^2 u(y, z) = C,$$

where $\tilde{\nabla}^2$ is the two-dimensional Laplace operator in the $(y, z)$-plane, and $C$ is a constant. Show how $C$ is related to the fluid pressure and the fluid’s dynamic viscosity, $\mu$.

(b) We will now use the result derived in part (a) to determine the pressure-driven flow in an infinitely long pipe whose cross-section is given by the equilateral triangle shown in the figure below.

(i) State the boundary conditions for $u(y, z)$ and explain the rationale behind choosing the ansatz

$$u(y, z) = A(y - h)(y^2 - 3z^2)$$

where $A$ is a constant. [Note the orientation of the coordinate axes.]

(ii) Determine the constant $A$ for the case when the flow is driven by an imposed pressure gradient $\partial p/\partial x = G$ where the constant $G$ is given.

[15 marks]
4. A Newtonian incompressible fluid of density $\rho$ and kinematic viscosity $\nu$ occupies the region between two semi-infinite planes, $\varphi = \alpha$ and $\varphi = -\alpha$, where $(r, \varphi, z)$ are cylindrical polar coordinates with the axis $r = 0$ located at the intersection of the two planes at point A in the sketch shown below. A uniform line sink along this axis draws fluid towards the axis at a constant rate $Q$ (volume per unit time per unit length of the axis). You may assume that the velocity field is steady and two-dimensional, and takes the form $\mathbf{u} = u(r, \varphi) \mathbf{e}_r + v(r, \varphi) \mathbf{e}_\varphi$. [Hint: The Navier-Stokes equations in cylindrical polar coordinates are given on the formula sheet at the end of the exam paper.]

(a) Use dimensional arguments to show that $u$ must be proportional to $1/r$. [Hint: The dimensional units of $\nu$ are the same as those of $Q$, namely $m^2/sec$.]

(b) Given the result from part (a), use the continuity equation and the boundary conditions to show that $v \equiv 0$.

(c) Show that if the radial velocity is expressed in the form

$$u = \frac{\nu}{r} f(\varphi),$$

then for the two momentum equations to be consistent with this form of the solution, the pressure must take the form

$$p = \rho \left( \frac{2\nu^2}{r^2} f(\varphi) + \frac{K}{r^2} \right),$$

where $K$ is a constant.

(d) Show that the problem can be reduced to the solution of the ODE

$$f''' + 4f' + 2ff' = 0$$

and state the three boundary conditions to be satisfied by $f(\varphi)$. 

[25 marks]
5. The figure below shows a stationary solid cylinder of radius \( a \) that is located in a viscous, incompressible fluid of infinite extent. Far away from the cylinder the velocity field approaches a uniform shear flow such that

\[
\mathbf{u} \to S \ y \ \mathbf{e}_x \quad \text{as} \quad r = \sqrt{x^2 + y^2} \to \infty
\]

where \( S \) is a given constant and \( r \) the radial distance from the centre of the cylinder.

(a) Using cylindrical polar coordinates, state the velocity boundary conditions to be applied on the surface of the cylinder and show that the far-field boundary condition stated above implies that

\[
\left( \begin{array}{c}
\frac{\partial u_r}{\partial r} \\
\frac{\partial u_\varphi}{\partial r}
\end{array} \right) \to \frac{1}{2} S \ r \ \left( \begin{array}{c}
\sin(2\varphi) \\
\cos(2\varphi) - 1
\end{array} \right) \quad \text{as} \quad r \to \infty.
\]

[Hint: You may wish to use the following identities: \( \sin(\varphi) \cos(\varphi) = \frac{1}{2} \sin(2\varphi) \); \( \sin^2(\varphi) = \frac{1}{2} (1 - \cos(2\varphi)) \) and \( \cos^2(\varphi) = \frac{1}{2} (1 + \cos(2\varphi)) \). The formula sheet at the end of the exam paper lists useful relations between cartesian and cylindrical polar coordinates.]

(b) Assuming that the flow is steady, two-dimensional and sufficiently slow for Stokes’ equations to apply, state the partial differential equation for the streamfunction \( \psi(r, \varphi) \). Explain briefly why it is sensible to look for a solution in the form \( \psi(r, \varphi) = F(r) + G(r) \cos(2\varphi) \).

(c) Using the solution of the biharmonic equation in cylindrical polar coordinates, provided on the formula sheet at the end of the exam paper, find the streamfunction.
Some useful equations in plane cylindrical polar coordinates:

- **Relation to cartesian coordinates:**
  \[ x = r \cos(\varphi) \quad \text{and} \quad y = r \sin(\varphi) \]
  \[ e_x = e_r \cos(\varphi) - e_\varphi \sin(\varphi) \quad \text{and} \quad e_y = e_r \sin(\varphi) + e_\varphi \cos(\varphi) \]
  where \( e_r \) and \( e_\varphi \) are unit vectors in the radial and azimuthal directions, respectively.

- **Velocity field:**
  \[ \mathbf{u} = u(r, \varphi, t) \mathbf{e}_r + v(r, \varphi, t) \mathbf{e}_\varphi. \]

- **Navier Stokes equations**
  \[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \varphi} - \frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[ \nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \varphi} \right], \]
  \[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \varphi} + \frac{uv}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \varphi} + \nu \left[ \nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} \right], \]
  \[ \text{div} \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial \varphi} = 0, \]

- **The Laplace operator**
  \[ \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}. \]

- **The streamfunction**
  \[ u = \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial r}. \]

- **The general form of the solution of the biharmonic equation, \( \nabla^4 \psi(r, \varphi) = 0 \), in cylindrical polars can be represented by superposition of the following solutions:**
  - The general axisymmetric solution:
    \[ \psi(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \]
    \[ \psi(r, \varphi) = \left( A r + \frac{B}{r} + C r^3 + Dr \ln r \right) \cos(\varphi) \]
    \[ + \left( a r + \frac{b}{r} + c r^3 + dr \ln r \right) \sin(\varphi) \]
    \[ \psi(r, \varphi) = \sum_{n=2}^{\infty} \left( A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2} \right) \cos(n\varphi) \]
    \[ + \left( a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2} \right) \sin(n\varphi). \]

The coefficients \((A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1, a_1, b_1, c_1, d_1, A_2, B_2, C_2, D_2, a_2, b_2, c_2, d_2, ...)\) have to be determined from the boundary conditions.

**END OF EXAMINATION PAPER**