Two Hours

THE UNIVERSITY OF MANCHESTER

APPLIED COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS

Friday, 24th January 2014
14:00 – 16:00

Answer ALL FOUR questions in SECTION A (45 marks in total) and Answer TWO of the THREE questions in SECTION B (40 marks in total). If more than two questions from Section B are attempted, then credit will be given for the best two answers.

Electronic calculators are permitted, provided they cannot store text.
SECTION A

Answer ALL FOUR questions

A1. The function \( f_1(z) \) is given by

\[
f_1(z) = (z + \sqrt{3})^{1/2} \ln(z - 1).
\]

The branch of this function is chosen such that

\[
-\frac{4\pi}{3} < \arg(z - 1) \leq \frac{2\pi}{3}, \quad \text{and} \quad -\frac{\pi}{2} < \arg\left(z + \sqrt{3}\right) \leq \frac{3\pi}{2}.
\]

Draw a clearly labelled diagram showing the branch cuts, and the polar coordinates used to evaluate the function, and express the function in terms of these coordinates. Find also the residues, at both the poles, of the function

\[
g(z) = f_1(z)/(z^2 + 1);
\]

leaving your answers in the form \( 2^{-a} e^{ib} (\ln 2 - ic) \), where \( a, b \) and \( c \) are all real.

[11 marks]

A2. By considering the contour integral around a ‘D-contour’ of the function

\[
f_2(z) = \frac{ze^{iaz}}{z^2 + 6z + 25}
\]

for \( a > 0 \), show that

\[
\int_{-\infty}^{\infty} \frac{x \sin(ax)}{x^2 + 6x + 25} \, dx = \frac{\pi}{4} e^{-4a} \left\{4 \cos(3a) + 3 \sin(3a)\right\}.
\]

[12 marks]

A3. Use the strong form of Schwartz’s reflection principle to find the function \( f_3(z) \) that has the following properties, stating carefully any results you use:

(i) \( f_3(z) \) is regular in \( \text{Im}(z) > 0 \), and continuous in \( \text{Im}(z) \geq 0 \), except for a simple pole at \( z = 2 + i \) with residue \( 4 - 5i \);

(ii) \( f_3(z) \) is real for real \( z \);

(iii) \( f_3(z) \to 2 \) as \( |z| \to \infty \) in \( \text{Im}(z) \geq 0 \).

[11 marks]

A4. Define the Laplace Transform \( F_4(p) \) of the function \( f_4(t) \), and state the formula for the inverse transform (Bromwich’s Integral).

For the particular case in which

\[
F_4(p) = \frac{6}{(p + 1)^2 + 9}
\]

find \( f_4(t) \) by contour integration. (You should describe the case \( t > 0 \) fully, but also give brief details for the case \( t < 0 \).)

[11 marks]
SECTION B

Answer TWO OF THE THREE questions

B5. The function $f_5(z)$ is defined by

$$f_5(z) = \frac{z^\alpha}{(z+1)^3},$$

where the parameter $\alpha$ is first considered to be real and in the interval $(-1, 2)$ and where the branch of the function is defined by $0 \leq \text{arg}(z) < 2\pi$.

(i) State the formula for the residue of a pole of order $n$ at $z = z_0$ and use it to find the residue of $f_5(z)$ at its only pole.

(ii) By an integration of $f_5(z)$ around a suitable closed (keyhole) contour, show that, for $\alpha$ in the given range,

$$\int_0^\infty \frac{x^\alpha}{(x+1)^3} \, dx = \frac{\pi\alpha(1-\alpha)}{2\sin(\pi\alpha)} \quad \text{(*)}.$$

(Your solution must include a clear, labelled diagram showing the contour, the branch cut and the position of the pole.)

(iii) Now consider $\alpha$ to be complex; use analytic continuation to show that (*) holds in a certain domain $D$ (which you should find) of the complex $\alpha$-plane.

[20 marks]

B6. (i) The Gamma function $\Gamma(z)$ is defined in the usual way by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt.$$

State the domain in which this integral defines a regular function of $z$, and derive the values of $\Gamma(1)$ and $\Gamma(1/2)$, for which you may assume without proof that $\int_0^\infty e^{-x^2} \, dx = \sqrt{\pi}$.

(ii) Show that

$$\Gamma(z+1) = z \Gamma(z)$$

in the above domain of regularity, and explain how this formula is used to produce an analytic continuation of $\Gamma(z)$ into the whole plane except for simple poles at the points $z = 0, -1, -2, -3, \ldots$. Find the residues of $\Gamma(z)$ at these poles.

(iii) By consideration of its singularities, show that the function $k(z)$ given by

$$k(z) = \Gamma\left(\frac{1}{2} + z\right) \Gamma\left(\frac{1}{2} - z\right) \cos(\pi z)$$

is an entire function. Under the additional assumption that $k(z)$ is bounded as $|z| \to \infty$, show that $k(z)$ is constant and find its value. Now use this formula to deduce that $\Gamma(z)$ never vanishes, and that, for real $y$,

$$\left| \Gamma\left(\frac{1}{2} + iy\right) \right|^2 = \frac{\pi}{\cosh(\pi y)}.$$ 

[20 marks]
B7. (a) Integrate the function $h(z) = e^{-m z^2}$, where $m$ is real and positive, around the rectangular contour with vertices at the points $z = \pm X$ and $\pm X + ic$ to derive the result, in the limit as $X \rightarrow \infty$,

$$\int_{-\infty}^{\infty} e^{-m x^2} \cos(2m c x) \, dx = \sqrt{\frac{\pi}{m}} e^{-m c^2}.$$ (\*)

[You may assume without proof that $\int_{-\infty}^{\infty} e^{-m x^2} \, dx = \sqrt{\frac{\pi}{m}}$ for $m > 0$.] [7 marks]

(b) Define the complex Fourier Transform $G(k)$ of the function $g(x)$ and write down the formula for the inverse transform.

The temperature $u(x, t)$ in an infinite straight uniform bar satisfies the heat-conduction equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad (\text{for } -\infty < x < \infty \text{ and } t > 0),$$

where $\alpha$ is a real positive constant. Initially the temperature is given by

$$u(x, 0) = \begin{cases} u_0 & (|x| < p), \\
0 & (|x| > p). \end{cases}$$

(Here $u_0$ and $p$ are real constants with $p > 0$.)

Derive the differential equation and initial condition satisfied by the complex Fourier Transform $U(k, t)$ of the temperature $u(x, t)$, and hence show that for $t > 0$

$$\frac{\partial U}{\partial x} = \frac{u_0}{(4\pi \alpha t)^{\frac{1}{2}}} \left( e^{-(x+p)^2/4\alpha t} - e^{-(x-p)^2/4\alpha t} \right).$$

[You may need to use the formula $2 \sin X \sin Y = \cos(X - Y) - \cos(X + Y)$; you will also need the displayed result \* from part \(a\) of this question.] [13 marks]