Two hours

THE UNIVERSITY OF MANCHESTER

GROUP THEORY

22 January 2014
2.00 - 4.00

Answer THREE of the four questions. If more than three questions are attempted, credit will be given for the best three answers.

Electronic calculators may be used, provided that they cannot store text.
1.

(i) Let $S$ be a non-empty subset of the group $G$.

(a) Prove that $\langle S \rangle$ is a subgroup of $G$.
(b) Prove that $\langle S \rangle = \bigcap_{S \subseteq H, H \subseteq G} H$.
(c) Let $G = A_5$. Prove that $\langle (1, 2)(3, 4), (1, 2, 3, 4, 5) \rangle = G$. (State clearly any properties of $A_5$ you use.)

(ii) Suppose that $H$ and $K$ are subgroups of the group $G$.

(a) If $HK = KH$, prove that $HK$ is a subgroup of $G$.
(b) Assume that $G$ is a finite group, and let $k_1, \ldots, k_n$ be a complete set of right coset representatives for $K \cap H$ in $K$. Prove that $HK = \bigcup_{i=1}^n Hk_i$ and that this union is disjoint. Hence deduce that $|HK| = \frac{|H||K|}{|H \cap K|}$.

[20 MARKS]

2.

(i) Define what is meant by "$G$ acts on $\Omega$", where $G$ is a group and $\Omega$ is a non-empty set.

(ii) Suppose that $G$ is a group acting upon the non-empty set $\Omega$.

(a) Prove that $\Omega$ is the disjoint union of its $G$-orbits.
(b) For $\alpha \in \Omega$, prove that $G_\alpha$ is a subgroup of $G$.
(c) Assume that $G = \langle g_1, g_2, g_3 \rangle \leq S_{20}$ where

$$g_1 = (1, 5)(2, 6)(4, 9)(3, 11)(10, 20)(18, 19)(14, 16)$$
$$g_2 = (1, 10, 5)(4, 6, 8)(7, 11, 12)(13, 16, 14) \quad \text{and}$$
$$g_3 = (2, 19)(4, 6, 17, 18)(11, 12, 13, 16)(15, 20).$$

Determine the $G$-orbits on $\Omega = \{1, \ldots, 20\}$.

(iii) Let $G$ be a finite group which acts on a finite non-empty set $\Omega$. If $G$ has $t$ orbits on $\Omega$, prove that

$$t = \frac{1}{|G|} \sum_{g \in G} |\text{fix}_\Omega(g)|.$$  

(You may suppose without proof, that $|G| = |\Delta||G_\alpha|$ where $\Delta$ is a $G$-orbit of $\Omega$ and $\alpha \in \Delta$.)

[20 MARKS]
3.

(i) (a) State the classification theorem for finitely generated abelian groups.
(b) List up to isomorphism all abelian groups of order $p^4$, where $p$ is a prime.

(ii) (a) State the Jordan-Hölder theorem for finite groups.
(b) Give composition series for $S_4$ and $S_5$, explaining why they are composition series. Also give the composition factors for $S_4$ and $S_5$.

(iii) Suppose that $G$ is a group. Prove that if $G/Z(G)$ is cyclic, then $G$ is abelian.

[20 MARKS]

4. Suppose that $G$ is a finite group.

(i) Assume that $|G| = p^r m$ where $p$ is a prime and $p$ does not divide $m$. Prove that $G$ contains at least one subgroup of order $p^r$. (You may assume, without proof, that a finite group whose order is divisible by $p$ has at least one element of order $p$.)

(ii) Suppose that $|G| = pq$ where $p$ and $q$ are distinct primes such that $p$ does not divide $q - 1$. Prove that $G$ has a normal Sylow $p$-subgroup.

(iii) Suppose that $|G| = pq$ where $p$ and $q$ are primes such that $p < q$ and $p$ does not divide $q - 1$. Prove that $G$ is a cyclic group.

In parts (ii) and (iii) state clearly which parts of Sylow's theorem you use.

[20 MARKS]