Two hours

THE UNIVERSITY OF MANCHESTER

DIFFERENTIABLE MANIFOLDS

17 January 2014
14:00 – 16:00

Answer ALL FOUR questions in Section A (40 marks in total).
Answer TWO of the THREE questions in Section B (60 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

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Electronic calculators are not allowed.

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Throughout the paper, where the index notation is used, the Einstein summation convention over repeated indices is applied if it is not explicitly stated otherwise.
SECTION A

Answer ALL FOUR questions

A1.

(a) For a set $X$, explain what is meant by a chart (or local coordinate system) on $X$ and what is meant by an atlas on $X$.

(b) Define the notion of a change of coordinates between two charts and explain what is meant by a smooth atlas.

(c) Consider the real projective line $\mathbb{RP}^1$ as the set of equivalence classes $[(x^1 : x^2)]$ of non-zero vectors $(x^1, x^2) \in \mathbb{R}^2 \setminus \{0\}$ with respect to the equivalence relation $(x^1 : x^2) = (\lambda x^1 : \lambda x^2)$ for arbitrary $\lambda \neq 0$. Let $U_i = \{(x^1 : x^2) \mid x^i \neq 0\} \subset \mathbb{RP}^1$, $i = 1, 2$. Define maps $\varphi_i: \mathbb{R} \to U_i$ by $\varphi_1(u) = (1, u)$ and $\varphi_2(v) = (v, 1)$. Each $\varphi_i$ is a chart and the collection $(\varphi_i)_{i=1,2}$ is an atlas for $\mathbb{RP}^1$. [You do not have to show this.] Calculate the change of coordinate $\varphi_{12} = \varphi_1^{-1} \circ \varphi_2$. [10 marks]

A2.

(a) Give the definition of a tangent vector $u$ at a point $x \in M$ in terms of the transformation law for its components $u^i$.

(b) Define the tangent space $T_x M$. Define the vector operations (addition and multiplication by scalars) for the elements of $T_x M$. [You do not have to show that the operations are well-defined.]

(c) For a smooth curve $x = x(t)$ in $M$ define its velocity $\dot{x}$ and show that for all $t$ the velocity $\dot{x}$ is an element of the tangent space $T_{x(t)} M$. [10 marks]

A3.

(a) Explain what is meant by the commutator of vector fields on a manifold. [You do not have to show that the space of vector fields is closed under commutator.] Give a coordinate expression for the commutator of vector fields $u = u^i e_i$ and $w = w^j e_j$. (Here $(e_i)$ is the basis of tangent vectors associated with a local coordinate system, so that $e_i = \frac{\partial}{\partial x^i}$.)

(b) Consider the vector fields on $\mathbb{R}^3$

$$P_1 = \frac{\partial}{\partial x}, \quad P_2 = \frac{\partial}{\partial y}, \quad L = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$  

Calculate their pairwise commutators, that is: $[P_i, P_j]$ and $[L, P_i]$, $i = 1, 2$, expressing the answer in terms of $P_i$ and $L$ themselves.
A4.

(a) Give the axioms defining the *exterior differential*

\[ d: \Omega^p(M) \to \Omega^{p+1}(M), \]

for all \( p = 0, 1, 2, \ldots, n \), on a manifold \( M^n \).

(b) Deduce from the axioms the coordinate expression for the differential \( d\sigma \) of a 1-form \( \sigma \) on a 2-dimensional manifold, where

\[ \sigma = \sigma^1 dx^2 - \sigma^2 dx^1 \]

(here \( \sigma^1, \sigma^2 \) are given functions).

(c) Explain what is meant by saying that a form \( \omega \) on a manifold \( M \) is *closed* and what is meant by saying that it is *exact*. Show that the 1-form \( \omega = d\theta \) on \( \mathbb{R}^2 \setminus \{0\} \), where \( \theta \) is the polar angle, is closed but not exact.
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SECTION B

Answer TWO of the THREE questions

B5.

(a) Explain what is meant by a closed submanifold of dimension $k$ of a smooth manifold $M^n$.
Consider a subset $\Gamma \subset \mathbb{R}^n$ specified by an equation $F(x^1, \ldots, x^n) = 0$ such that $dF(x) \neq 0$ for all $x = (x^1, \ldots, x^n) \in \Gamma$. Show that for every $x \in \Gamma$ there exist local coordinates $y^1, \ldots, y^n$ defined in some open neighborhood $W$ of $x$ such that $\Gamma \cap W$ is specified by the equation $y^n = 0$.

(b) Explain what is meant by saying that a system of equations

$$f^\mu(x) = 0$$

(1)

in $\mathbb{R}^n$, where $\mu = 1, \ldots, k$, has constant rank.
State without proof the theorem about the solution set of a system of equations of constant rank.
Explain the practical method of showing that a system (1) has constant rank using the auxiliary linear system

$$\frac{\partial f^\mu}{\partial x^i}(x_0) v^i = 0,$$

(2)

where $x_0$ satisfies the original system $f^\mu(x) = 0$.

(c) Consider the subset $S \subset \mathbb{R}^3 \times \mathbb{R}^3$ consisting of all pairs $(a, b)$ of vectors such that $|a| = |b| = 1$ and $a \cdot b = 0$ (we use the standard scalar product on $\mathbb{R}^3$). Show that the set $S$ has a natural structure of a manifold of dimension 3. Hint: to analyze the auxiliary linear system, choose a unit vector $n$ orthogonal to $a$ and $b$ and expand the velocities of curves in $S$ over the orthonormal basis $a, b, n$.

[30 marks]

B6.

(a) Explain what is meant by a derivation over an algebra homomorphism $\alpha: A_1 \to A_2$, where $A_1$ and $A_2$ are commutative associative algebras.
State Hadamard’s lemma in $\mathbb{R}^n$ (you do not have to give a proof) and deduce from it that, for arbitrary $x_0 \in \mathbb{R}^n$, all derivations $D: C^\infty(\mathbb{R}^n) \to \mathbb{R}$ over the evaluation homomorphism $ev_{x_0}: C^\infty(\mathbb{R}^n) \to \mathbb{R}$ have the form $D = \partial_v$ for some $v \in \mathbb{R}^n$.

(b) Show that the commutator of vector fields satisfies the Jacobi identity.

(c) Consider on the unit circle $S^1$ with the standard polar coordinate $\theta$ the complex-valued vector fields

$$L_n = e^{-in\theta} \frac{d}{d\theta},$$

where $n \in \mathbb{Z}$. Find the commutator $[L_n, L_m]$ for arbitrary $n$ and $m$ and show that it can be expressed as a linear combination with constant coefficients of the vector fields $L_k$, $k \in \mathbb{Z}$.
B7.

(a) Explain what is meant by a partition of unity on a manifold $M$ and when a partition of unity $(f_\alpha)$ is said to be subordinate to an open cover $\mathcal{U} = (U_\alpha)$.

For a compact manifold $M$, show the existence of a finite partition of unity subordinate to an arbitrary open cover. [You do not need to prove the existence of bump functions.]

(b) Give the definition of the integral of an $n$-form $\omega$ over a compact oriented manifold $M^n$.

Prove that the integral $\int_M \omega$ does not depend on choices of an atlas (with a given orientation) and a partition of unity. [You may assume without proof that the integral over a single coordinate domain does not depend on a choice of coordinates as long as orientation is fixed.]

(c) Calculate the integral of the 1-form on $\mathbb{R}^2$

$$\omega = r^\alpha (xdy - ydx)$$

over a circle $x^2 + y^2 = R^2$. Here $x$ and $y$ are standard coordinates, $r^2 = x^2 + y^2$, and $\alpha$ is a parameter. For which values of $\alpha$ the integral does not depend on the radius $R$?

[30 marks]