Two hours

THE UNIVERSITY OF MANCHESTER

FINANCIAL MATHEMATICS FOR ACTUARIAL SCIENCE II

15 January 2014
09.45 – 11.45

Answer FIVE of the SIX questions. If more than FIVE questions are attempted, then credit will be given for the best FIVE answers.

Electronic calculators may be used, provided that they cannot store text.
1. A company is considering whether to invest in one of the following projects.
Project A: For an investment of £100,000 the company will receive £27,000 at the start of each of the next eight years.
Project B: For an immediate investment of £120,000 followed by further expenditure of £2000 at the end of the first three years of the project, the company will receive payments of £135000 at the end of the sixth, seventh and eighth years.

   (a) Show that the yields for each project are 21.2% and 18.5% respectively, each correct to one decimal place.  

   [8 marks]

   (b) Find the net present value of each project to the nearest pound if the effective annual interest rate is 15%.

   [8 marks]

   (c) Without further calculation, which project would you recommend, and why would you make this recommendation, with effective annual interest rate of 15%? What if the effective annual interest rate was 20%?

   [4 marks]
2. (a) An engineering company has agreed to supply a large manufacturer with a certain number of parts for each of the next seven years. It estimates that this contract will involve an immediate outlay of £100,000 to set up a plant, and that there will be further production costs of £80,000 each year, incurred at the end of each of the seven years of the project. For ecological reasons they are also committed to paying clean-up fees at the end of the contract of £40,000 p.a. payable annually for three years with the first payment due in eight years time. Sales of the parts are expected to generate £120,000 p.a. paid annually for seven years with the first payment in one year’s time.

The project is to be financed by a loan from the bank charging interest on loans at an effective annual interest rate of 15% and the loan can be paid off by income at any time. The bank pays 12% effective annual interest on deposits.

(i) When is the bank account expected to be in credit for the first time?

(ii) What balance, to the nearest pound, does the company expect to have in the bank at the end of the clean-up period (i.e. after the full ten years of the project)?

(b) During the year 2012-2013 an investment fund was valued quarterly with the results (in £millions)

<table>
<thead>
<tr>
<th>1 January 2012</th>
<th>1 April 2012</th>
<th>1 July 2012</th>
<th>1 October 2012</th>
<th>1 January 2013</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.14</td>
<td>0.92</td>
<td>1.03</td>
<td>1.30</td>
<td>1.55</td>
</tr>
</tbody>
</table>

The cashflows during this period were:
- 31 March 2012: pay out £100,000 to an investor closing her account;
- 30 June 2012: sell shares for £50,000, which is re-invested in the fund;
- 30 September 2012: receive £20,000 interest on investments which is re-invested in the fund; and
- 30 September 2012: receive £150,000 from a new investor.

(i) Find the annual time weighted rate of return for the year 2012 as a percentage correct to 2 decimal places. If the payments and valuation of 31 March/1 April had occurred at the end of February/beginning of March how would this figure change?

(ii) Find the annual money weighted rate of return for the year 2012 to the nearest 1%.
3. (a) Write down the cashflow associated with a $n$-year fixed interest security having coupons $D$ per unit nominal payable annually *in advance*, so the first coupon is paid at the time the security is purchased. Use this cashflow to show that if the investor pays income tax at $t_x\%$ at source, is not liable for capital gains tax, and the redemption price is $R$ per unit nominal then the issue price $P$ per unit nominal to obtain a yield $i$ is

$$P = \frac{(1 - t_x)D}{i}(1 + i - (1 + i)^{-n+1}) + R(1 + i)^{-n}.$$  

[10 marks]

(b) A loan of nominal £10,000 is to be redeemed at 110% at the end of ten years. The loan bears interest (coupons) at effective rate of 5% payable annually *in advance*. An investor paying income tax (deducted at source) at 40% and capital gains tax at 40% (paid at the end of the loan on the difference between the redemption and issue prices when this is positive). The investor wishes to purchase the entire loan on the issue date at a price to obtain an effective yield of 8% per annum. Will capital gains tax be payable? What price should be paid for the loan, correct to the nearest pound.?  

[10 marks]

4. (a) An asset pays coupons at 10% of its nominal value every 13 months starting on the 1 January 2014, ending with a last coupon payment on 1 July 2019 after which the dividend will be re-calculated according to the prevailing market conditions. What forward price per unit nominal should be agreed on 1 July 2014 to buy the asset on 1 July 2017 if the force of interest is $\delta$% p.a. and the price of the asset on 1 July 2014 is $S_0$ per unit nominal? State carefully any results used.  

[10 marks]

(b) Define the $n$–year spot rates $y_n$ of a given market and the forward rates $f_{m,n}$. Suppose the price of a zero coupon bond with term $n$ is

$$P_1 = 0.98, \quad P_2 = 0.94, \quad P_4 = 0.88,$$

$n = 1, 2, 4$. Find the spot rate $y_1$, $y_2$ and $y_4$ and as many of the forward rates $f_{m,n}$, $m \geq 1$, $n \geq 1$, as you can from this data, giving answers as percentages correct to two decimal places.  

[10 marks]
5. (a) Define the effective duration of a cashflow in terms of its net present value, \( NPV(i) \). Two discrete cashflows have net present values \( N_j \) and effective durations \( V_j \), \( j = 1, 2 \). Show from the definition that if \( N_1 + N_2 \neq 0 \) then the effective duration of the sum of the cashflows is

\[
\frac{N_1 V_1 + N_2 V_2}{N_1 + N_2}.
\]

[10 marks]

(b) An investor will receive an annuity of £20,000 p.a. annually in arrears for ten years with the first payment in one year’s time. The investor has outstanding liabilities for two payments of £60,000 each, one due in five years’ time and the other ten years later. The effective interest rate is 10% p.a..

(i) What is the net present value and the effective duration of the investor’s income and what is the net present value and effective duration of the investor’s liabilities.

[8 marks]

(ii) Use the result of part (a) to determine the effective duration of the investor’s net cashflow.

[2 marks]

6.

(a) In each of the next five years the effective annual interest rate will be a constant, chosen independently each year from the set \( \{5\%, 7\%, 10\%\} \) with probability \( (\frac{1}{3}, \frac{1}{2}, \frac{1}{6}) \) respectively. An investor requires £10,000 in five years time. What should be invested now so that the expected value of the investment is at least this amount in five years’ time? What is the probability, expressed as a percentage correct to one decimal place, that the shortfall will be more than 5% of the amount required?

[10 marks]

(b) An investor requires £1,000,000 in 50 years’ time and has decided to make an immediate investment of £10,000. Assuming that the effective annual accumulation rate of the fund, \( (1 + i_t) \), is log-normally distributed with mean \( \mu = 0.1 \) and standard deviation \( \sigma = 0.15 \). Show that the expectation of this investment after 50 years exceeds the required sum. Find the probability that the investor will meet the required target (assuming that the annual accumulation rates are independent), leaving your answer in the form \( \Phi(x) \) where \( \Phi \) is the accumulation function of the \( N(0, 1) \) distribution, with \( x > 0 \) to be determined.

[10 marks]