Two and a half hours

THE UNIVERSITY OF MANCHESTER

SETS, NUMBERS AND FUNCTIONS B

23 January 2014
9.45 - 12.15

Answer **ALL SIX** questions in Section A (40 marks in total). Answer **FOUR** of the SIX questions in Section B (60 marks in total). If more than **FOUR** questions from Section B are attempted, then credit will be given for the best **FOUR** answers.

Electronic calculators may be used, provided that they cannot store text.
SECTION A

Answer **ALL** of the SIX questions

**A1.**
For statements \( p \) and \( q \), determine the truth tables of the following:

(a) \( p \lor q \);

(b) \( (p \land \neg q) \Rightarrow ((\neg q) \lor (\neg p)) \).

[6 marks]

**A2.** Let \( f : A \rightarrow B \) be a function.

(a) Define \( \text{Im}(f) \), the image of \( f \).

(b) Define what it means for \( f \) to be 1-1.

(c) Define what it means for \( f \) to be onto.

(d) Define what it means for \( f \) to be a permutation.

[7 marks]

**A3.** Let \( R \) be a relation on a set \( A \).

(a) Define what it means for \( R \) to be symmetric.

(b) Define what it means for \( R \) to be reflexive.

(c) Define what it means for \( R \) to be transitive.

(d) Give an example of a set \( A \) and a relation \( R \) on \( A \) which is transitive, but not symmetric or reflexive.

[8 marks]

**A4.** Use induction to show that 5 divides \( 6^n + 4 \) for all \( n \in \mathbb{N} \)

[7 marks]
A5. Consider the permutation

\[ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 8 & 7 & 2 & 6 & 5 & 9 & 1 & 3 \end{pmatrix}. \]

(a) Write down \( f \) in disjoint cycle notation.
(b) Write down \( f^{-1} \), using any notation you wish.
(c) Let \( a = (19258)(3674) \) and \( b = (154)(296)(38) \). Write down \( a \circ b \) as a product of disjoint cycles.

[6 marks]

A6. Let \( * \) be a binary operation on a non-empty set \( S \). Explain what it means for

(a) \( * \) to be associative;
(b) \( * \) to be commutative;
(c) \( e \) to be an identity element of \( S \) for \( * \).

[6 marks]
SECTION B

Answer **FOUR** of the SIX questions.

B7.

(a) Define a binary operation $*$ on $\mathbb{Z}$ by

$$a * b = ab + 1$$

where $a, b \in \mathbb{Z}$.

(i) Determine whether or not $*$ is commutative, justifying your answer.

(ii) Show that $*$ is not associative.

(iii) Determine whether or not $\mathbb{Z}$ possesses an identity element with respect to $*$, justifying your answer.

[5 marks]

(b) Let $G$ be a non-empty set and $*$ be a binary operation on $G$. Define what it means for $(G, *)$ to be a group.

[4 marks]

(c) Suppose $(G, *)$ is a group.

(i) Prove that the identity element of $(G, *)$ is unique.

(ii) Let $e$ be the identity element of $(G, *)$. If $g, h \in G$, then prove that $(gh)^{-1} = h^{-1}g^{-1}$.

[6 marks]

B8.

(a) Using the Euclidean algorithm, find the greatest common divisor of 102 and 174, and express this greatest common divisor as an integer linear combination of 102 and 174.

[6 marks]

(b) Find all integers $x \in \{0, 1, \ldots, 173\}$ which are solutions of

$$102x \equiv 12 \mod 174.$$ 

[6 marks]

(c) Consider the group $(\mathbb{Z}_{28}, \circ)$. Find the inverse of the element 12.

[3 marks]

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B9.

(a) Let $A$ be a non-empty set and $R$ an equivalence relation on $A$.

(i) For $a \in A$, define the equivalence class $R_a$ of $a$.
(ii) If $a, b \in A$ and $aRb$, prove that $R_a = R_b$.
(iii) If $a, b \in A$ and $a \not R b$, prove that $R_a \cap R_b = \emptyset$.

[9 marks]

(b) Let $A$ and $B$ be sets and let $f : A \rightarrow B$ be a function. Define an equivalence relation $R$ on $A$
by
\[ a_1Ra_2 \iff f(a_1) = f(a_2), \]
where $a_1, a_2 \in A$.

(i) Give a necessary and sufficient condition on the equivalence classes for $f$ to be 1-1.

[2 marks]

(ii) Suppose now that $A = B = \mathbb{Z}_{12}$ and define $f : A \rightarrow A$ by $f(x) = 4 \odot x$, i.e., multiplication
modulo 12. Write down the equivalence classes of $R$ in this case.

[4 marks]

B10.

(a) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions. Prove that if $g \circ f$ is 1-1, then $f$ is 1-1.

[4 marks]

(b) Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by
\[ f(x) = \begin{cases} \frac{x+3}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} \]
Show that $f$ is both 1-1 and onto.

[8 marks]

(c) Determine whether the following statement is true or false, and give a proof of either the
statement or its negation as appropriate:
\[ \exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \forall z \in \mathbb{Z}, z \leq x + y + 1. \]

[3 marks]
B11.

(a) Let $A$ and $B$ be sets.

(i) Define the Cartesian product $A \times B$.

(ii) Let $A = \{(n, 2n) : n \in \mathbb{Z}\}$ and $B = \{(n, n + 1) : n \in \mathbb{Z}\}$ (both subsets of $\mathbb{Z} \times \mathbb{Z}$). Write down $A \setminus B$. [3 marks]

(b) Prove that every natural number greater than 1 has a prime divisor. [7 marks]

(c) Show that the equation $x^2 + 8y = 3$ has no integer solutions $x$ and $y$. [5 marks]

B12.

(a) Let $A$ be a set and let $n \in \mathbb{N}$. Define what it means for $A$ to have cardinality $n$. [2 marks]

(b) Using the cardinality of appropriate sets, define the binomial coefficient \( \binom{n}{r} \), where $n, r \in \mathbb{N} \cup \{0\}$. [3 marks]

(c) Using the above definition of \( \binom{n}{r} \), prove that if $n, r \in \mathbb{N} \cup \{0\}$ with $n \geq r$, then
\[
\binom{n}{r} = \binom{n}{n - r}.
\] [4 marks]

(d) State (without proof) the binomial theorem. [3 marks]

(e) Prove that
\[
\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n.
\] [3 marks]

END OF EXAMINATION PAPER.

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