Throughout the paper you may assume that the Dirichlet Convolution of two multiplicative functions is multiplicative.

SECTION A

Question 1. i. Show, by estimating integrals or otherwise, that

\[ \int_1^N \frac{du}{u^\sigma} + \frac{1}{N^\sigma} \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} < 1 + \int_1^N \frac{du}{u^\sigma}, \]

for real \( \sigma > 0 \).

Deduce that the series defining \( \zeta(\sigma) \) diverges for \( \sigma \leq 1 \), converges for \( \sigma > 1 \) and satisfies

\[ \frac{1}{\sigma-1} \leq \zeta(\sigma) \leq 1 + \frac{1}{\sigma-1} \]

for \( \sigma > 1 \).

ii. Explain why

\[ \sum_{n \in \mathcal{N}} \frac{1}{n^\sigma} = \prod_{p \leq N} \left(1 - \frac{1}{p^\sigma}\right)^{-1}, \]

for \( N \geq 1 \), where \( \mathcal{N} = \{ n : p|n \Rightarrow p \leq N \} \).

iii Prove that

\[ \log \zeta(\sigma) \leq 1 + \sum_p \frac{1}{p^\sigma}, \]

for \( \sigma > 1 \).

Deduce that there are infinitely many primes.

You may assume that \( \sum_p (- \log (1 - 1/p^\sigma) - 1/p^\sigma) \leq 1 \) for \( \sigma \geq 1 \).

[30 marks]
Solution 1

i Use

\[
\text{glb}_\left[a, b\right] f(t) (b - a) \leq \int_a^b f(t) \, dt \leq \text{lub}_\left[a, b\right] f(t) (b - a),
\]

with \( f(t) = 1/t^\sigma \), a **decreasing** function. So first, with \([a, b] = [n, n + 1]\),

\[
\int_n^{n+1} \frac{dt}{t^\sigma} \leq \frac{1}{n^\sigma} ((n + 1) - n) = \frac{1}{n^\sigma}.
\]

Sum over \( n = 1, 2, \ldots, N - 1 \) to get

\[
\int_1^N \frac{dt}{t^\sigma} + \frac{1}{N^\sigma} \leq \sum_{1 \leq n \leq N} \frac{1}{n^\sigma}.
\]

Second, with \([a, b] = [n - 1, n]\),

\[
\int_{n-1}^{n} \frac{dt}{t^\sigma} \geq \frac{1}{n^\sigma} (n - (n - 1)) = \frac{1}{n^\sigma}.
\]

Sum over \( n = 2, \ldots, N \) to get

\[
\sum_{1 \leq n \leq N} \frac{1}{n^\sigma} \leq \frac{1}{1^\sigma} + \int_1^N \frac{dt}{t^\sigma}.
\]

Combine to get

\[
\int_1^N \frac{du}{u^\sigma} + \frac{1}{N^\sigma} \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} < 1 + \int_1^N \frac{du}{u^\sigma}. \tag{1}
\]

**Bookwork** [9 marks]

It is possible to prove this by Partial Summation which I will accept.

Recall, from the definition of convergence of a series, that \( \zeta(\sigma) \) converges if, and only if

\[
\lim_{N \to \infty} \sum_{1 \leq n \leq N} \frac{1}{n^\sigma} \text{ exists.}
\]

If \( \sigma = 1 \) the left hand side of (1) gives a lower bound on the partial sum of

\[
\sum_{1 \leq n \leq N} \frac{1}{n^\sigma} \geq \frac{1}{N^\sigma} + \log N.
\]
The right hand side here $\to \infty$ as $N \to \infty$ in which case the series defining $\zeta(\sigma)$ diverges.

If $\sigma \neq 1$ then evaluating the integrals in (1) gives

$$\frac{N^{1-\sigma} - 1}{1 - \sigma} + \frac{1}{N^\sigma} \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} \leq 1 + \frac{N^{1-\sigma} - 1}{1 - \sigma}.$$  

If $\sigma < 1$ then $1 - \sigma > 0$ and so $N^{1-\sigma} \to \infty$ as $N \to \infty$ in which case, by the left hand inequality, a lower bound on the partial sums, the series defining $\zeta(\sigma)$ diverges.

If $\sigma > 1$ then $1 - \sigma < 0$ and so $N^{1-\sigma} \to 0$ as $N \to \infty$ in which case, by the right hand inequality, an upper bound on the partial sums, the series defining $\zeta(\sigma)$ converges.

We also get, for $\sigma > 1$, in the limit,

$$-\frac{1}{1 - \sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq 1 - \frac{1}{1 - \sigma},$$

equivalent to stated result.  

ii. Recalling that $(1 - y)^{-1}$ is the sum of the geometric series $1 + y + y^2 + y^3 + \ldots$, we get

$$\prod_{p \leq N} \left(1 - \frac{1}{p^\sigma}\right)^{-1} = \prod_{p \leq N} \left(1 + \frac{1}{p^\sigma} + \frac{1}{p^{2\sigma}} + \frac{1}{p^{3\sigma}} + \frac{1}{p^{4\sigma}} + \ldots\right).$$  \hspace{1cm} (2)

On multiplying out the Euler product we get terms $1/n^\sigma$ with integers $n$ composed only of primes $\leq N$, i.e. $n \in \mathcal{N}$.

Conversely, by the factorisation of integers into primes, every integer in $\mathcal{N}$ will arise and, by the unique factorisation of integers into primes, every integer in $\mathcal{N}$ will occur only once. Hence stated result.

Bookwork [5 marks]

iii. Start from

$$\zeta(\sigma) - \prod_{p \leq N} \left(1 - \frac{1}{p^\sigma}\right)^{-1} = \sum_{n \notin \mathcal{N}} \frac{1}{n^\sigma}.$$  

Assume $n \leq N$. Let $p | n$, $p$ prime, in which case $p \leq n \leq N$. That is, $p | n \Rightarrow p \leq N$, which is the definition of $n \in \mathcal{N}$. Hence $n \leq N \Rightarrow n \in \mathcal{N}$. So

$$0 \leq \sum_{n \notin \mathcal{N}} \frac{1}{n^\sigma} \leq \sum_{n \geq N+1} \frac{1}{n^\sigma} \leq \int_{N}^{\infty} \frac{dt}{t^\sigma} = \frac{N^{1-\sigma}}{\sigma - 1}.$$
Thus

\[ 0 \leq \zeta(\sigma) - \prod_{p \leq N} \left(1 - \frac{1}{p^\sigma}\right)^{-1} \leq \frac{N^{1-\sigma}}{(\sigma - 1)} \]  

(3)

If \( x > y > 0 \) and \( 0 < x - y < M \) then

\[ \log x - \log y = \int_y^x \frac{dt}{t} \leq \frac{1}{y} \int_y^x dt = \frac{x - y}{y} \leq \frac{M}{y}. \]

With \( x = \zeta(\sigma) \) and \( y = \prod_{p \leq N} (1 - p^{-\sigma})^{-1} \) we get

\[ 0 \leq \log \zeta(\sigma) - \log \prod_{p \leq N} \left(1 - \frac{1}{p^\sigma}\right)^{-1} \leq \varepsilon(N) \]  

(4)

where

\[ \varepsilon(N) = \frac{M}{y} = \frac{N^{1-\sigma}}{(\sigma - 1)} \prod_{p \leq N} \left(1 - \frac{1}{p^\sigma}\right) \leq \frac{N^{1-\sigma}}{(\sigma - 1)}, \]

since all factors in the product are \(< 1\). This bound tends to 0 as \( N \to \infty \) since \( \sigma > 1\).

Next

\[ \log \prod_{p \leq N} \left(1 - \frac{1}{p^\sigma}\right)^{-1} = - \sum_{p \leq N} \log \left(1 - \frac{1}{p^\sigma}\right) \]

\[ = \sum_{p \leq N} \left(- \log \left(1 - \frac{1}{p^\sigma}\right) - \frac{1}{p^\sigma}\right) + \sum_{p \leq N} \frac{1}{p^\sigma} \]

\[ \leq 1 + \sum_{p \leq N} \frac{1}{p^\sigma}, \]

by the assumption given in the question. Combine to get

\[ 0 \leq \log \zeta(\sigma) < 1 + \sum_{p \leq N} \frac{1}{p^\sigma} + \varepsilon(N). \]

Let \( N \to \infty \) to get

\[ 0 \leq \log \zeta(\sigma) \leq 1 + \sum_p \frac{1}{p^\sigma}. \]  

(5)

Bookwork [5 marks]
Combining parts i and iii gives

\[ \sum_p \frac{1}{p^\sigma} \geq \log \left( \frac{1}{\sigma - 1} \right) - 1, \]

for \( \sigma > 1 \). Let \( \sigma \to 1^+ \). The right hand side diverges as thus must the series on the left hand side. Yet all terms in the series remain finite so there must be infinitely many terms, i.e. infinitely many primes.  

Bookwork [3 marks]
Feedback on Question 1

i. You can use Euler Summation, i.e.

\[
\sum_{1 \leq n \leq N} f(n) = \int_1^N f(t) \, dt + f(1) + \int_1^N \{t\} f'(t) \, dt,
\]

for integer \(N\). You have to be careful with the final term. You cannot simply say \(|\{t\}| \leq 1\) or \(\{t\} = O(1)\), you have to use \(0 \leq \{t\} \leq 1\).

Problems arose if the inequalities

\[
\int_n^{n+1} \frac{dt}{t^\sigma} \leq \frac{1}{n^\sigma} \quad \text{and} \quad \int_{n-1}^n \frac{dt}{t^\sigma} \geq \frac{1}{n^\sigma}
\]

were summed over \(n = 1, ..., N\). The resulting integrals were then over 1 to \(N + 1\) or 0 to \(N\).

I should have written the next part of the question as

“Deduce that the series defining \(\zeta(\sigma)\)

- diverges for \(\sigma \leq 1\),
- converges for \(\sigma > 1\) and
- satisfies

\[
\frac{1}{\sigma - 1} \leq \zeta(\sigma) \leq 1 + \frac{1}{\sigma - 1},
\]

for \(\sigma > 1\)” because too many students forgot to prove at least one of these parts.

iii. The proof given above for (3) came after, in the lecture notes, the proof that \(\zeta(s) = \prod_p (1 - p^{-s})^{-1}\) for complex \(s\). Students tried to give a proof of (3) using complex \(s\) in place of the real \(\sigma\). This led to unsatisfactory confusion.

If we let \(N \to \infty\) in (3) we get \(\zeta(\sigma) = \prod_p (1 - p^{-\sigma})^{-1}\) for \(\sigma > 1\). Some students assumed this and continued with

\[
\log \zeta(\sigma) = \log \prod_p \left(1 - \frac{1}{p^\sigma}\right)^{-1} = \sum_p \log \left(1 - \frac{1}{p^\sigma}\right)^{-1}
\]

\[
= \sum_p \left(-\log \left(1 - \frac{1}{p^\sigma}\right) - \frac{1}{p^\sigma}\right) + \sum_p \frac{1}{p^\sigma}
\]

\[
\leq 1 + \sum_p \frac{1}{p^\sigma},
\]

(6)
by the assumption given in the question. But not only has this ‘proof’ started with an assumption it has also used the fact that the logarithm of an infinite product equals the infinite sum of logarithms. Statements about interchanging an operation, such as the taking of logarithms with infinite sums and products require proofs, usually by truncating the sums and products as in the proof given in the solutions.

A different proof of (6) can be given, based on the idea that the log function is continuous (in the form \( \lim_{n \to \infty} \log (a_n) = \log (\lim_{n \to \infty} a_n) \)).

Starting from \( n \leq N \Rightarrow n \in \mathcal{N} \), we have

\[
\sum_{1 \leq n \leq N} \frac{1}{n^\sigma} \leq \sum_{n \in \mathcal{N}} \frac{1}{n^\sigma} = \prod_{p \leq N} \left( 1 - \frac{1}{p^\sigma} \right)^{-1},
\]

by part ii. Take the logarithm to get

\[
\log \left( \sum_{1 \leq n \leq N} \frac{1}{n^\sigma} \right) \leq \log \left( \prod_{p \leq N} \left( 1 - \frac{1}{p^\sigma} \right)^{-1} \right) = \sum_{p \leq N} - \log \left( 1 - \frac{1}{p^\sigma} \right) = \sum_{p \leq N} \frac{1}{p^\sigma} + \sum_{p \leq N} \left( - \log \left( 1 - \frac{1}{p^\sigma} \right) - \frac{1}{p^\sigma} \right).
\]

Since \( \sigma > 1 \) we can now let \( N \to \infty \) when all the three series will converge, (using the assumption in the question on the last sum). In detail

\[
\log (\zeta (\sigma)) = \log \left( \lim_{N \to \infty} \sum_{1 \leq n \leq N} \frac{1}{n^\sigma} \right) = \lim_{N \to \infty} \log \left( \sum_{1 \leq n \leq N} \frac{1}{n^\sigma} \right)
\]

here we are using that log is continuous,

\[
= \lim_{N \to \infty} \sum_{p \leq N} \frac{1}{p^\sigma} + \lim_{N \to \infty} \sum_{p \leq N} \left( - \log \left( 1 - \frac{1}{p^\sigma} \right) - \frac{1}{p^\sigma} \right)
\]

\[
= \sum_{p} \frac{1}{p^\sigma} + \sum_{p} \left( - \log \left( 1 - \frac{1}{p^\sigma} \right) - \frac{1}{p^\sigma} \right) \leq \sum_{p} \frac{1}{p^\sigma} + 1.
\]
Again using the assumption in the question at the last step.
Question 2. i. By Partial Summation prove that for $s \neq 1$ we have

$$\sum_{1 \leq n \leq N} \frac{1}{n^s} = 1 + \frac{1}{s-1} + \frac{N^{1-s}}{1-s} - s \int_1^N \{u\} \frac{du}{u^{s+1}}$$

for any integer $N \geq 1$.

ii. Deduce that

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \{u\} \frac{du}{u^{1+s}},$$

(7)

for Re $s > 1$.

Explain why (7) can be used to define $\zeta(s)$ for complex $s$ with Re $s > 0$, $s \neq 1$.

iii) Using parts i and ii prove that for all complex $s$ with Re $s > 0$, $s \neq 1$, and all integers $N \geq 1$,

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + O\left(\frac{|s|}{\sigma N^\sigma}\right).$$

Deduce that

$$\sum_{n=1}^\infty \frac{1}{n^{1+it}}$$

diverges for all real $t > 0$.

[30 marks]
Solution 2

i. By Partial Summation

\[ \sum_{1 \leq n \leq N} \frac{1}{n^s} = \sum_{1 \leq n \leq N} \left( \frac{1}{N^s} - \left( \frac{1}{N^s} - \frac{1}{n^s} \right) \right) \]

\[ = \frac{N}{N^s} - \sum_{1 \leq n \leq N} \int_{n}^{N} (-s) \frac{du}{u^{s+1}} \]

\[ = \frac{N}{N^s} + s \int_{1}^{N} \left( \sum_{1 \leq n \leq u} 1 \right) \frac{du}{u^{s+1}} \]

\[ = \frac{N}{N^s} + s \int_{1}^{N} [u] \frac{du}{u^{s+1}}. \]

Thus

\[ \sum_{1 \leq n \leq N} \frac{1}{n^s} = \frac{N}{N^s} + s \int_{1}^{N} [u] \frac{du}{u^{s+1}} \]

\[ = \frac{N}{N^s} + \frac{s}{1 - s} (N^{1-s} - 1) - s \int_{1}^{N} \{u\} \frac{du}{u^{s+1}}, \]

which rearranges to stated result.

ii. We have Re\( s > 1 \) so \( 1 - \sigma < 0 \). Thus

\[ \left| \frac{N^{1-s}}{1 - s} \right| = \frac{N^{1-\sigma}}{|1 - s|} \to 0 \text{ as } N \to \infty. \]

Also the resulting integral satisfies

\[ \int_{1}^{\infty} \left| \{u\} \right| \frac{du}{u^{s+1}} \leq \int_{1}^{\infty} \frac{du}{u^{\sigma+1}} = \frac{1}{\sigma}, \]

i.e. it converges (absolutely). So we can let \( N \to \infty \) to get the stated result

\[ \zeta(s) = 1 + \frac{1}{s - 1} - s \int_{1}^{\infty} \{u\} \frac{du}{u^{1+s}}, \]

for Re\( s > 1 \).
Looking at (8) we see that the integral in fact converges for $\sigma > 0$. This is why the right hand side of (9) can be used to define a function on $\text{Re } s > 0$ which agrees with the series definition of $\zeta (s)$ on $\text{Re } s > 1$.

iii Subtract the last two results to get, for $\text{Re } s > 0$

$$\zeta (s) - \sum_{1 \leq n \leq N} \frac{1}{n^s} = -\frac{N^{1-s}}{1-s} + s \int_1^N \{u\} \frac{du}{u^{s+1}} - s \int_1^\infty \{u\} \frac{du}{u^{1+s}} \quad \text{i.e.}$$

$$\zeta (s) = \sum_{1 \leq n \leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^\infty \frac{du}{u^{1+s}}.$$

The integral here is estimated as

$$\left| s \int_N^\infty \frac{\{u\} \ du}{u^{1+s}} \right| \leq \left| s \right| \left| \int_N^\infty \frac{du}{u^{1+s}} \right| = \frac{|s|}{\sigma N^\sigma}.$$

With $s = 1 + it$, $t > 0$, the last result rearranges to

$$\sum_{1 \leq n \leq N} \frac{1}{n^{1+it}} = \zeta (1 + it) + \frac{e^{i(t \log N + \pi/2)}}{t} + O \left( \frac{|t|}{N} \right).$$

As $N \to \infty$ we get a sequence of partial sums that get ever closer to a circle, centre $\zeta (1 + it)$ and radius $1/t$, and keep going round the circle without end. Hence we do not have convergence to a point and so must have divergence.
Feedback on Question 2

i. I also allowed students to use Euler Summation, if they remembered it correctly as

\[ \sum_{1 \leq n \leq x} f(n) = \int_1^x \frac{f(t)}{n} dt + f(1) - \{x\} f(x) + \int_1^x \{t\} f'(t) dt. \]

(The danger of trying to remember such a result, instead of deriving it as needed, is getting the signs wrong.)

ii. When \( a_n \) are complex to show \( a_n \to 0 \) as \( n \to \infty \) it suffices to show \( |a_n| \to 0 \). I had statements like ‘1 - s is negative hence \( N^{1-s} \to 0 \). This does not make sense.

Too many students forgot to show that the completed integral \( \int_1^\infty \{u\} u^{-s-1} du \) converges for \( \text{Re} \, s > 1 \). You can only \( N \to \infty \) in the result of part i when you know that the resulting integral exists. Strange, for many of these students then went on to show the integral converges for \( \text{Re} \, s > 0 \), thus giving the continuation of \( \zeta(s) \) to that larger half plane.

iii Some students did not tell me what they were doing, i.e. subtracting the results of part i and ii.
Questions 3. i. Write down the Euler product for the Riemann zeta function $\zeta(s)$.

ii Let $\omega(n)$ denote the number of distinct prime factors of $n$. By looking at the Euler product of the Dirichlet Series on the left hand side of the identity below, prove that

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)},$$

for Re $s > 1$.

*You may assume that $2^\omega$ is multiplicative.*

iii Let $\lambda$ be Liouville’s function, defined as $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime divisors of $n$ counted with multiplicity. By looking at the Euler product of the Dirichlet Series on the left hand side of the identity below, prove that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)},$$

for Re $s > 1$.

*You may assume that $\lambda$ is multiplicative.*

iv. Explain why parts ii and iii suggest that

$$2^\omega * \lambda = 1.$$

Prove this by showing equality on prime powers.

[30 marks]
Solution 3. i

\[ \zeta(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}, \]

for \( \text{Re} \ s > 1. \)

ii. Using the fact that \( 2^\omega \) is multiplicative and \( 2^\omega(p^a) = 2 \) for all primes \( p \) and \( a \geq 1, \)

\[ \sum_{n=1}^{\infty} \frac{2^\omega(n)}{n^s} = \prod_p \left( 1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \frac{2}{p^{3s}} + \ldots \right). \]

Sum the series

\[ 1 + 2y + 2y^2 + 2y^3 + \ldots = 1 + 2y \left( 1 + y + y^2 + \ldots \right) = 1 + \frac{2y}{1 - y} \]

\[ = \frac{1 + y}{1 - y} = \frac{1 - y^2}{(1 - y)^2}. \]

Applying this with \( y = 1/p^s \) gives

\[ \prod_p \left( 1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \frac{2}{p^{3s}} + \ldots \right) = \prod_p \left( \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^s}} \right)^2 = \frac{\zeta^2(s)}{\zeta(2s)}, \]

by part i.

iii. Using the fact that \( \lambda \) is multiplicative and \( \lambda(p^a) = (-1)^a \) for all primes \( p \) and \( a \geq 1, \)

\[ \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \left( 1 - \frac{1}{p^s} + \frac{1}{p^{2s}} - \frac{1}{p^{3s}} + \ldots \right). \]

This time each sum is a geometric series,

\[ 1 - y + y^2 - y^3 + \ldots = \frac{1}{1 - (-y)} = \frac{1}{1 + y} = \frac{1 - y}{1 - y^2}. \]

Hence

\[ \prod_p \left( 1 - \frac{1}{p^s} + \frac{1}{p^{2s}} - \frac{1}{p^{3s}} + \ldots \right) = \prod_p \left( \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^{2s}}} \right) = \frac{\zeta(2s)}{\zeta(s)}. \]
iv With \( D_f(s) := \sum_{n=1}^{\infty} f(n) n^{-s} \), parts ii and iii give

\[
D_{2^w \ast \lambda}(s) = D_{2^w}(s) D_{\lambda}(s) = \frac{\zeta^2(s) \zeta(2s)}{\zeta(2s) \zeta(s)} = \zeta(s) = D_1(s)
\]

for \( \text{Re } s > 1 \).

We know that if \( f = g \) then \( D_f(s) = D_g(s) \). We have never shown that \( D_f(s) = D_g(s) \) for an appropriate set of \( s \) implies \( f = g \), we have only ever said it ‘suggests’ \( f = g \). Thus \( D_{2^w \ast \lambda}(s) = D_1(s) \) suggests \( 2^w \ast \lambda = 1 \).

Since \( 2^w, \lambda \) and thus \( 2^w \ast \lambda \) are multiplicative

\[
2^w \ast \lambda(n) = 2^w \ast \lambda \left( \prod_{p^r|n} \right) = \prod_{p^r|n} 2^w \ast \lambda(p^r).
\]

Yet, by the definition of Dirichlet Convolution,

\[
2^w \ast \lambda(p^r) = \sum_{\substack{a+b=r \\ a,b \geq 0}} 2^w p^a \lambda(p^b).
\]

We take out the \( a = 0 \) separately for \( 2^w(p^a) = 2^0 = 1 \), so

\[
\sum_{\substack{a+b=r \\ a,b \geq 0}} 2^w p^a \lambda(p^b) = \lambda(p^r) + 2 \sum_{0 \leq b \leq r-1} \lambda(p^b)
\]

\[
= (-1)^r + 2 \sum_{0 \leq b \leq r-1} (-1)^b.
\]

This sum is a finite geometric sum with common ratio \(-1\). Thus

\[
2^w \ast \lambda(p^r) = (-1)^r + 2 \frac{1 - (-1)^r}{1 - (-1)} = 1,
\]

as required.
Feedback for Question 3

i. I wanted to see $\text{Re } s > 1$.

ii. Students should have said why the Dirichlet series can be written as an Euler product, i.e. because the arithmetic function is multiplicative.

iv. I wanted to see a reference to $D_{2^\omega * \lambda} (s) = D_{2^\omega} (s) D_\lambda (s)$. Many students multiplied $D_{2^\omega} (s)$ and $D_\lambda (s)$ but didn’t say why this was giving information on $2^\omega * \lambda$.

In the proof above we came to

$$2^\omega * \lambda (p^r) = (-1)^r + 2 \sum_{0 \leq b \leq r-1} (-1)^b.$$

Most students proceeded by examining two cases, $r$ odd and $r$ even.

If $r$ is odd then there are an odd number of term in the sum. We pair up these terms to get

$$2^\omega * \lambda (p^r) = -1 + 2 ((1 - 1) + (1 - 1) + ... + (1 - 1) + 1)$$
$$= -1 + 2 = 1.$$

If $r$ is even then there are an even number of term in the sum. We pair up these terms to get

$$2^\omega * \lambda (p^r) = 1 + 2 ((1 - 1) + (1 - 1) + ... + (1 - 1))$$
$$= 1 + 0 = 1.$$
Question 4

i. State, without proof, Möbius Inversion, not forgetting to define all terms.

ii a. Define Euler’s phi function, $\phi$.

b. Using Mobius Inversion or otherwise prove that $\phi = \mu * j$, i.e.

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d},$$

for all $n \geq 1$. Here $j$ is the identity function, $j(n) = n$ for all $n$.

Deduce that

c. 

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

for all $n \geq 1$.

d. 

$$\sum_{d|n} \phi(d) = n,$$

for all $n \geq 1$.

iii Prove that

$$\sum_{n \leq x} \frac{\phi(n)}{n} = \frac{1}{\zeta(2)} x + O(\log x).$$

You may assume that $\sum_{n>x} 1/n^2 = O(1/x)$ and $\sum_{n\leq x} 1/n = O(\log x)$.

[30 marks]
Solution 4 i Möbius Inversion states that

\[ \mu \ast 1 = \delta \quad \text{or equivalently,} \quad \sum_{d|n} \mu(d) = \delta(n). \]

Here \( \delta(n) = 1 \) for all \( n \geq 1 \) while \( \delta(n) = 0 \) for all \( n \geq 2 \). If \( n = \prod_{i=1}^{r} p_i^{a_i} \) is a factorization into distinct primes then the Möbius function is

\[ \mu(n) = \begin{cases} (-1)^r & \text{if } a_1 = a_2 = a_3 = \ldots = 1, \\ 0 & \text{if some } a_i \geq 2. \end{cases} \]

ii.a. Euler’s phi function is \( \phi(n) = |\{1 \leq r \leq n : \gcd(r,n) = 1\}| \), i.e.

\[ \phi(n) = \sum_{\substack{1 \leq r \leq n \\ \gcd(r,n) = 1}} 1. \]

b. Rewrite the condition \( \gcd(r,n) = 1 \) in terms of \( \delta \) as

\[ \phi(n) = \sum_{1 \leq r \leq n} \delta(\gcd(r,n)) = \sum_{1 \leq r \leq n} \sum_{d|\gcd(r,n)} \mu(d), \]

by Möbius Inversion. Note that \( d|\gcd(r,n) \) if, and only if, \( d|r \) and \( d|n \).

Interchange summations to get

\[ \phi(n) = \sum_{d|n} \mu(d) \sum_{1 \leq r \leq n \atop d|r} 1. \]

In the inner sum we can write \( r = sd, n = md \) and we are counting the number of integers \( s \leq m \), of which there are \( m = n/d \), hence

\[ \phi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = \sum_{d|n} \mu(d) \cdot \left( \frac{n}{d} \right) = (\mu \ast j)(n). \]

c) Since \( \mu \) and \( j \) are multiplicative then so is \( \phi \) (using the assumption given at the start of the paper). So it suffices to consider, with \( r \geq 1 \) and prime \( p \),

\[ \phi(p^r) = (\mu \ast j)(p^r) = \sum_{a+b=r \atop a,b \geq 0} \mu(p^a) j(p^b) = \mu(p^0) j(p^r) + \mu(p^1) j(p^{r-1}). \]
since \( \mu (p^a) = 0 \) if \( a \geq 2 \). Thus

\[
\phi (p^r) = p^r - p^{r-1} = p^r \left( 1 - \frac{1}{p} \right).
\]

Multiply together to get stated result.

**Bookwork [4 marks]**

d) By definition of Dirichlet Convolution

\[
\sum_{d|n} \phi (d) = (1 * \phi) (n).
\]

Yet

\[
1 * \phi = 1 * (\mu * j) \quad \text{by part b}
\]
\[
= (1 * \mu) * j
\]
\[
= \delta * j \quad \text{by Mobius Inversion}
\]
\[
= j \quad \text{since } \delta \text{ is the identity under *}
\]

Hence

\[
\sum_{d|n} \phi (d) = (1 * \phi) (n) = j (n) = n.
\]

**Problem Sheet [5 marks]**

I’ll accept other proofs, i.e. partitioning integers \( r \) depending on \( \gcd (r, n) \)

iii By Part ii b,

\[
\sum_{n \leq x} \frac{\phi (n)}{n} = \sum_{n \leq x} \sum_{d|n} \frac{\mu (d)}{d} = \sum_{d \leq x} \frac{\mu (d)}{d} \sum_{n \leq x} \frac{1}{d|n},
\]

on interchanging summations. Continuing

\[
= \sum_{d \leq x} \frac{\mu (d)}{d} \left[ \frac{x}{d} \right] = \sum_{d \leq x} \frac{\mu (d)}{d} \left( \frac{x}{d} + O (1) \right) = x \sum_{d \leq x} \frac{\mu (d)}{d^2} + O \left( \sum_{d > x} \frac{1}{d} \right).
\]

The error here is \( O (\log x) \) by assumption in question. In the main term complete the sum up to infinity

\[
\sum_{d \leq x} \frac{\mu (d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu (d)}{d^2} - \sum_{d > x} \frac{\mu (d)}{d^2} = \frac{1}{\zeta (2)} + O \left( \sum_{d > x} \frac{1}{d^2} \right).
\]
The error here is $O(1/x)$ by assumption in question. Combining

$$
\sum_{n \leq x} \frac{\phi(n)}{n} = x \left( \frac{1}{\zeta(2)} + O \left( \frac{1}{x} \right) \right) + O(\log x).
$$

Keeping only the dominant error term we get the stated result.

Bookwork [9 marks]
Feedback on Question 4

i. A third variant on Möbius inversion is if \( f \) and \( g \) are arithmetic functions then \( f = 1 * g \) if, and only if, \( g = \mu * f \).

ii. Some students noted that, by Möbius inversion, \( \phi = \mu * j \) iff

\[
1 * \phi = 1 * (\mu * j) = (1 * \mu) * j = \delta * j = j,
\]

i.e. \( 1 * \phi = j \). So to prove \( \phi = \mu * j \) they attempted to prove \( 1 * \phi = j \). To this end they proved \((1 * \phi)(p^r) = p^r\) for all prime powers. But how can you put these together?

To say

\[
(1 * \phi)(n) = (1 * \phi) \left( \prod_{p^a || n} p^a \right) = \prod_{p^a || n} (1 * \phi)(p^a) = \prod_{p^a || n} p^a = n
\]

you need to know that \( 1 * \phi \) is multiplicative. We have the result that if \( f \) and \( g \) are multiplicative then so is \( f * g \) but to apply this you would need to know that \( \phi \) is multiplicative. Yet, until we show that \( \phi = \mu * j \), we do not know \( \phi \) is multiplicative. Thus this method cannot be used.
SECTION B

This Section is Compulsory, answer all parts.

Question 5. i. a) Prove that there exists a constant \( \gamma \) such that

\[
\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O \left( \frac{1}{x} \right),
\]

for real \( x > 1 \).

b) Explain why this error term is best possible for real \( x \).

c) Prove that

\[
\sum_{n \leq N} \frac{1}{n} = \log N + \gamma + \frac{1}{2N} + O \left( \frac{1}{N^2} \right),
\]

for integer \( N \geq 1 \).

You may assume that \( \psi_2 (x) := \int_0^x (\{t\} - 1/2) \, dt \) is periodic in \( x \), with period 1.

ii. The Bernoulli polynomials and numbers are defined iteratively by

\[
P_k (x) = k \int_0^x P_{k-1} (t) \, dt + B_k \quad \text{for } k \geq 2,
\]

where each \( B_k \) is chosen so that

\[
\int_0^1 P_k (t) \, dt = 0,
\]

along with \( P_1 (x) = \{x\} - 1/2 \) when \( x \notin \mathbb{Z} \), 0 when \( x \in \mathbb{Z} \).

a) Find the Fourier Series for \( P_k (x) \), \( k \geq 2 \).

You may assume that every \( P_k (x) \) is periodic with period 1 and \( P_1 (x) \) has Fourier Series \(-\sum_{n \neq 0} e^{2\piinx} / (2in\pi)\).

b) Deduce that

\[
\zeta (2\ell) = \frac{(-1)^{\ell+1} (2\pi)^{2\ell}}{2 (2\ell)!} B_{2\ell},
\]

for all \( \ell \geq 1 \). [45 marks]
Solution 5

i. a. From either Partial Summation or, as here, from Euler Summation with \( f(x) = \frac{1}{x} \) we have

\[
\sum_{n \leq x} \frac{1}{n} = \int_1^x \frac{dt}{t} + 1 - \frac{\{x\}}{x} - \int_1^x \frac{\{t\}}{t^2} \, dt. \tag{10}
\]

The second integral converges absolutely since

\[
\int_1^\infty \frac{\{|t|\}}{t^2} \, dt \ll \int_1^\infty \frac{dt}{t^2} = 1.
\]

Thus we can complete the integral up to \( \infty \), the error in doing so is

\[
\leq \int_x^\infty \frac{\{|t|\}}{t^2} \, dt \ll \int_x^\infty \frac{dt}{t^2} \ll \frac{1}{x}.
\]

Combining,

\[
\sum_{n \leq x} \frac{1}{n} = \log x + 1 + O\left(\frac{1}{x}\right) - \int_1^\infty \frac{\{t\}}{t^2} \, dt.
\]

Hence the result follows with

\[
\gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2} \, dt.
\]

b. The error is best possible in that as \( x \) moves from \( N^- \) to \( N^+ \) (where \( N \) is an integer) we gain a term \( 1/N \) in the sum \( \sum_{n \leq x} 1/n \), whereas, because of continuity, the main terms \( \log x + \gamma \) vary by almost nothing. Hence the error term has to accommodate, i.e. be no less than, the \( 1/N \), which is approximately \( 1/x \)

Bookwork \[8 \text{ marks}\]

\[3 \text{ marks}\]

c. Return to (10) with \( x = N \), and integer. Then

\[
\sum_{n \leq N} \frac{1}{n} = \int_1^N \frac{dt}{t} + 1 - \int_1^N \frac{\{t\}}{t^2} \, dt.
\]

Write

\[
\int_1^N \frac{\{t\}}{t^2} \, dt = \frac{1}{2} \int_1^N \frac{dt}{t^2} + \int_1^N \frac{\{t\}}{t^2} \, dt - \frac{1}{2} \int_1^N \frac{dt}{t^2}
\]

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The first integral equals
\[ \frac{1}{2} \left(1 - \frac{1}{N}\right). \]

For the second integral, integration by parts gives
\[
\int_1^N \frac{\{t\} - 1/2}{t^2} dt = \left[ \frac{\psi_2(t)}{t^2} \right]_1^N + 2 \int_1^N \frac{\psi_2(t)}{t^3} dt.
\]

By definition \(\psi_2(0) = 0\). We are told that \(\psi_2(t)\) is periodic, period 1 and so \(\psi_2(N) = \psi_2(0) = 0\) for all integers \(N\). Thus
\[
\left[ \frac{\psi_2(t)}{t^2} \right]_1^N = 0,
\]
and
\[
\int_1^N \frac{\{t\} - 1/2}{t^2} dt = 2 \int_1^N \frac{\psi_2(t)}{t^3} dt. \tag{11}
\]

Since \(\psi_2\) is periodic and continuous (being defined by an integral) it is bounded. Hence the integral in (11) converges, so complete to infinity and bound the tail end as
\[
\int_N^\infty \frac{\psi_2(t)}{t^3} dt \ll \int_N^\infty \frac{dt}{t^3} \ll \frac{1}{N^2}.
\]

Therefore
\[
\sum_{n \leq N} \frac{1}{n} = \log N + C + \frac{1}{2N} + O\left(\frac{1}{N^2}\right), \tag{12}
\]
with
\[ C = \frac{1}{2} - 2 \int_1^\infty \frac{\psi_2(t)}{t^3} dt. \]

From (12)
\[
C = \lim_{N \to \infty} \left(\sum_{n \leq N} \frac{1}{n} - \log N - \frac{1}{2N}\right)
\]
\[ = \lim_{N \to \infty} \left(\sum_{n \leq N} \frac{1}{n} - \log N\right) = \gamma \]

by Part i.a
ii. a Since the Bernoulli functions $P_k(x)$ are periodic with period 1, they have a Fourier Series

$$
\sum_{n=-\infty}^{\infty} c_n(k) e^{2\pi inx} \quad \text{where} \quad c_n(k) = \int_0^1 P_k(x) e^{-2\pi inx} dx.
$$

From the definition of $P_k$ we have $\int_0^1 P_k(t) dt = 0$ which implies $c_0(k) = 0$ for all $k \geq 1$.

Assume $n \neq 0$. From the definition we have $P_k'(x) = kP_{k-1}(x)$ so integration by parts gives

$$
c_n(k) = \int_0^1 P_k(x) e^{-2\pi inx} dx
= \left[ -P_k(x) \frac{e^{-2\pi inx}}{2\pi in} \right]_0^1 + \frac{k}{2\pi in} \int_0^1 P_{k-1}(x) e^{-2\pi inx} dx
= \frac{k}{2\pi in} c_n(k-1).
$$

Continue,

$$
c_n(k) = k! \left( \frac{1}{2\pi in} \right)^{k-1} c_n(1).
$$

Next, by the given assumption,

$$
P_1(x) = -\sum_{n \neq 0} \frac{e^{2\pi inx}}{2in\pi} \quad \text{so} \quad c_n(1) = -\frac{1}{2in\pi}.
$$

Hence

$$
c_n(k) = -k! \left( \frac{1}{2\pi in} \right)^k.
$$

Thus, for $k \geq 1$,

$$
P_k(x) = -\frac{k!}{(2\pi i)^k} \sum_{n \neq 0} \frac{e^{2\pi inx}}{n^k}.
$$

ii. b. If we set $x = 0$ and recall $P_k(0) = B_k$ for $k \geq 2$, we get

$$
B_k = -\frac{k!}{(2\pi i)^k} \sum_{n \neq 0} \frac{1}{n^k}.
$$
In the sum we group $n$ and $-n$ together. For each such pair $n > 0$ and $-n$, we have
\[ \frac{1}{n^k} + \frac{1}{(-n)^k} = \frac{2}{n^k} \text{ if } k \text{ even, } 0 \text{ if } k \text{ is odd.} \]

Restricting to even $k = 2\ell$,
\[ B_{2\ell} = -\frac{(2\ell)!}{(2\pi i)^{2\ell}} 2 \sum_{n=1}^{\infty} \frac{1}{n^{2\ell}} = (-1)^{\ell+1} 2 \frac{(2\ell)!}{(2\pi)^{2\ell}} \zeta(2\ell). \]

This rearranges to the stated result. \text{Bookwork [6 marks]}
Feedback on Question 5

i.c. An alternative to the proof given of Part c is to start from

\[
\sum_{n \leq N} \frac{1}{n} = \int_{1}^{N} \frac{dt}{t} + 1 - \int_{1}^{N} \frac{\{t\}}{t^2} dt
\]

\[
= \log N + 1 - \int_{1}^{\infty} \frac{\{t\}}{t^2} dt + \int_{N}^{\infty} \frac{\{t\}}{t^2} dt.
\]

Write

\[
\int_{N}^{\infty} \frac{\{t\}}{t^2} dt = \frac{1}{2} \int_{N}^{\infty} \frac{dt}{t^2} + \int_{N}^{\infty} \frac{\{t\}}{t^2} + \frac{1}{2} dt.
\]

The first integral equals \(1/2N\). For the second integral, integration by parts gives

\[
\int_{N}^{\infty} \frac{\{t\}}{t^2} dt = \left[ \frac{\psi_2(t)}{t^2} \right]_{N}^{\infty} + 2 \int_{N}^{\infty} \frac{\psi_2(t)}{t^3} dt.
\]

Since \(\psi_2(t)\) is periodic, period 1 we have \(\psi_2(N) = \psi_2(0) = 0\) for all integers \(N\). But periodic also means \(\psi_2(t)\) is bounded in which case \(\lim_{t \to \infty} \psi_2(t)/t^2 = 0\) and thus

\[
\left[ \frac{\psi_2(t)}{t^2} \right]_{N}^{\infty} = 0.
\]

Hence

\[
\int_{N}^{\infty} \frac{\{t\}}{t^2} dt = 2 \int_{N}^{\infty} \frac{\psi_2(t)}{t^3} dt = O \left( \frac{1}{N^2} \right).
\]

The virtue of this method is that the constant \(C\) is immediately seen to be \(\gamma\).

With the proof of Part c given in the notes and repeated in the solutions above we find that

\[
C = \frac{1}{2} - 2 \int_{1}^{\infty} \frac{\psi_2(t)}{t^3} dt.
\]

A student noted that by letting \(N \to \infty\) in (11) we get

\[
2 \int_{1}^{\infty} \frac{\psi_2(t)}{t^3} dt = \int_{1}^{\infty} \frac{\{t\}}{t^2} dt - \frac{1}{2} + \int_{1}^{\infty} \frac{\{t\}}{t^2} dt\]

\[
= -\frac{1}{2} + (1 - \gamma),
\]

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since $\gamma = 1 - \int_{1}^{\infty} \{t\} t^{-2} dt$. Rearranging we find that $C = \gamma$.

ii When finding the Fourier Series for $P_k(x)$ I re-expressed the integral definition

$$P_k(x) = k \int_{0}^{x} P_{k-1}(t) \, dt + B_k$$

(13)

in terms of derivatives as $P'_k(x) = kP_{k-1}(x)$. This has the virtue of removing the $B_k$ term.

Some students attempted to use (13). For example, by using the Fourier Series for $P_1$ given in the question,

$$P_2(x) = 2 \int_{0}^{x} \left\{ \sum_{n\neq 0} \frac{e^{2\pi i nt}}{2in\pi} \right\} dt + B_2$$

$$= -2 \sum_{n\neq 0} \frac{1}{2in\pi} \left[ \frac{e^{2\pi i nt}}{2in\pi} \right]_{0}^{x} + B_2$$

$$= -2 \sum_{n\neq 0} \frac{e^{2\pi inx}}{(2\pi n)^2} + 2 \sum_{n\neq 0} \frac{1}{(2\pi n)^2} + B_2.$$  

(14)

You now have to recall that $B_2$ is chosen to make

$$\int_{0}^{1} P_2(t) \, dt = 0.$$  

But this condition means only that the Fourier Series for $P_2$ has 0 constant term. In terms of (14) this means

$$B_2 = -2 \sum_{n\neq 0} \frac{1}{(2\pi n)^2} \left( \frac{1}{\pi \zeta(2)} \right),$$

when

$$P_2(x) = -2 \sum_{n\neq 0} \frac{e^{2\pi inx}}{(2\pi n)^2}.$$  

Then students would need to guess the general form for $P_k$ (which makes this approach less satisfactory for it is not a proof). One thing to be careful of here is that you do not get a negative sign at each step.

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