A1 Part (i) is bookwork (see Lecture 3).

For the case of a straight line, the easiest way to do this is to start with the equation of a straight line in $\mathbb{R}^2$, namely $ax + by + c = 0$, write $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$, substitute in, and then simplify. Some of you worked backwards: starting with $\beta z + \bar{\beta}z + \gamma = 0$ and then re-arranging to get $ax + by + c = 0$, which is also fine. A small number of you started from $y = mx + c$: this equation doesn’t describe all straight lines in $\mathbb{R}^2$ as it misses out vertical straight lines.

For the case of a circle, the easiest starting point is the equation $|z - z_0| = r$, as in the lecture notes. You can start from the equivalent equation $(x - x_0)^2 + (y - y_0)^2 = r^2$ and then substitute for $x, y$ in terms of $z, \bar{z}$, at the expense of some algebraic manipulation.

Most of you who answered this question did well. Quite a few people only looked at the case of a straight line and didn’t do the circle case.

For part (ii) you can either do it by the safe way or the smart way. The safe way is to first note that $-5 + 12i$ and $12 + 5i$ don’t lie on a vertical straight line, hence $\alpha \neq 0$, and without loss of generality we can take $\alpha = 1$. Hence the points $z = -5 + 12i$ and $z = 12 + 5i$ both solve $z\bar{z} + \beta z + \bar{\beta}z + \gamma = 0$. This gives two equations in two unknowns: $169 - 10\beta + \gamma = 0$, $169 + 24\beta + \gamma = 0$. Solving these gives $\beta = 0, \gamma = -169$. Hence we have $z\bar{z} - 169 = 0$. (The smart way is to recognise that $-5 + 12i$ and $12 + 5i$ lie on the circle with centre 0 and radius 13—as $5, 12, 13$ is a Pythagorean triple—and the equation of such a circle is $|z| = 13$, i.e. $z\bar{z} - 169 = 0$. Just writing down ‘it’s a circle centre 0 and radius 13’ isn’t enough as the question asks you to find values of $\alpha, \beta, \gamma$.)

A2 Part (i) is a very standard definition from the course.

For part (ii), the intended way of doing it was to choose a parametrisation of the vertical straight line from $1 + 3i$ to $1 + 12i$ and then to use the definition in part (i) to calculate the length. Most of you did it this way. There are many different choices of parametrisation, and some of you used quite inventive ones - which is absolutely fine.

Some of you did this part of the question by arguing as follows: (i) Möbius transformations preserve the hyperbolic lengths of paths, (ii) the transformation $\gamma(z) = z - 1$ maps $1 + 3i$ and $1 + 12i$ and the vertical straight line between them to the $3i$ and $12i$ and the imaginary axis, (iii) the length of the arc of imaginary axis between $3i$ and $12i$ is $\log 12/3 = \log 4$. This is fine too.

It is not ok to say that the answer is $\log(1 + 12i)/(1 + 3i)$ (because it isn’t, indeed it isn’t even a real number).

A3 (i) The points I wanted you to make are: (i) the elements of $\Gamma$ are words of symbols, (ii) sub-words of the form $aa^{-1}$, $a^{-1}a$ are deleted, (iii) the group operation is concatenation, (iv) the empty word is the group identity, (v) the inverse of $a_{i_1}\cdots a_{i_n}$ is $a_{i_n^{-1}}\cdots a_{i_1^{-1}}$.

(ii) Let $S = \{a, b\}$. The words of length 1 are

$$a, b, a^{-1}, b^{-1}.$$ 

The words of length 2 are

$$a^2, ab, ab^{-1}, a^{-2}, a^{-1}b, a^{-1}b^{-1}, b^2, ba, ba^{-1}, b^{-2}, b^{-1}a^{-1}, b^{-1}a.$$
One way to see this is by trial-and-error and working out all the possibilities. Alternatively, note that the word has to start with either $a, a^{-1}, b, b^{-1}$. In each of these four cases there are three possible symbols that could come next to give a word of length 2 (a can be followed by $a, b, b^{-1}$ but not by $a^{-1}$ to give a word of length 2, etc). Hence there are $4 \times 3 = 12$ words of length 2.

This latter argument generalises. There are 4 words of length 1. If $w = a_{i_1} \cdots a_{i_n}$ is a word of length $n, n \geq 1$, then we obtain a word $a_{i_1} \cdots a_{i_n}a_{i_{n+1}}$ of length $n = 1$ provided $a_{i_{n+1}} \neq a_{i_n}^{-1}$. Hence there are 3 choices of $a_{i_{n+1}}$. By induction, there are $4 \times 3^{n-1}$ distinct words of length $n$.

Many of you got the words of length 1, most of you got the words of length 2, but the final part of the question was not, in general, well answered.

A4 There were some complaints that there was a mistake in the question as the diagram wasn’t labelled. There isn’t a mistake and you have all the information in the diagram that you need. You need to choose your own labelling of the vertices, edges and side-pairing transformations and the answers you get will not depend on this choice.

There are two elliptic cycles, call them $E_1$ (consisting of 9 vertices) and $E_2$ (consisting of just one vertex). The angle sum for $E_1$ is $9 \times \pi/9 = \pi$. The angle sum for $E_2$ is $\pi/9$.

The order of an elliptic cycle $E$ is the integer $m$ for which $m \times \text{sum}(E) = 2\pi$. Hence the order of $E_1$ is 2 and the order of $E_2$ is 18. A very significant number (probably over half of you) said that the order of $E_2$ is 9.

You can work out that the genus of $\mathbb{H}/\Gamma$ is 2 either by using the sketch in Lecture 22, or by using $2 - 2g = V - E + F$. The space $\mathbb{H}/\Gamma$ is a torus of genus 2 with 2 marked points, one of order 2 and one of order 18 (not 9).

B5 Very few of you attempted this question, and an even smaller number of you made a good attempt at it. The statements are of comparable difficulty to previous exam questions on fundamental domains; however, there is an extra intellectual step in that instead of being asked to prove/disprove something, here you first need to think whether the statement is true or false. I found some of your answers to these statements to be very useful as I’ve discovered some commonly-held misunderstandings about hyperbolic geometry that will enable me to improve the course in future years.

(i) False. The set isn’t open, so cannot be a fundamental domain. (Many of you didn’t get this.)

(ii) True. Take $\Gamma = \{\gamma_n \mid \gamma_n(z) = z + 4n\}$. (Those of you who attempted this question mostly got this right. Note that the question doesn’t ask you to construct a Dirichlet polygon so you don’t need to go through the lengthy algorithm to construct one. All you need do is to pick a group and then show that the claimed set is a fundamental domain for that group.)

(iii) False(*). See Figure 13.2.3 in the notes and the discussion. (*Although the question is slightly ambiguous. If $\Gamma = \{\text{id}\}$ is the trivial group then $F = \mathbb{H}$ is a fundamental domain and is the only fundamental domain for $\Gamma$. So in this case the statement is true. I should perhaps have said ‘Let $\Gamma$ be a non-trivial fundamental group.’)

(iv) True. This is a standard result from the course - see Proposition 14.3.1.

(v) False. The fundamental domain for the modular group constructed in Proposition 15.4.1 has hyperbolic area $\pi - (\pi/3 + \pi/3) = \pi/3$ by Gauss-Bonnet. By Proposition 13.2.1, all
fundamental domains for a given group have the same area. So there is no fundamental
domain for the modular group with area $\pi/8$.

(vi) True. See the example following Theorem 18.3.1.

(v) False. This was the only question on the paper that nobody answered correctly. The
trick is to think about rotations. Take a sector of the Poincaré disc with angle $2\pi \theta$ where
$\theta$ is irrational. This has two sides. If this is the fundamental domain for a Fuchsian
group $\Gamma$ then the side-pairing transformation must map one side to the other and fix
the origin; hence the side-pairing transformation must be a rotation through angle $2\pi \theta$.
As $\theta$ is irrational, this contradicts Theorem 17.2.1.

B6 Many of you gave good answers to this question, although (as usual with this kind of question)
some of you didn’t explain why, in (iii), applying a Möbius transformation to a right-angled
triangle didn’t change the angles.

(i) This is straightforward. Some of you had neat ways (quicker than the way I gave in the
solutions) to calculate $\text{Im}(\gamma(z))$.

(ii) Most of you who did this question got it right. (As an aside, any proof that starts
\[
\cosh d_H(z, w) = \cosh\{\inf\{\text{length}_{\mathbb{H}}(\sigma) \mid \cdots\}\}
\]
is never going to work. One simply observes that as $d_H(\gamma(z), \gamma(w)) = d_H(z, w)$ (as $\gamma$
is an isometry), applying $\cosh$ to both sides give $\cosh d_H(\gamma(z), \gamma(w)) = \cosh d_H(z, w)$.

(iii) This is a standard proof from the course. Remember that it’s not automatic that
isometries preserve angles - you need to point out that moving right-angled triangles
stay right-angled when applying Möbius transformations as Möbius transformations are
conformal.

(iv) I think all of you got the first bit right: split the quadrilateral into two right-angled
triangles.
For the ‘Euclidean’ bit: the resulting quadrilateral need not be a square (draw a picture).
The same argument for the hyperbolic case gives $a_1^2 + b_1^2 = a_2^2 + b_2^2$. Surprisingly few of
you did this bit or got it right.

B7 Many of you made very good attempts at this question gaining full or close-to-full marks.

(i) Showing that $z \mapsto (az + b)/(cz + d)$ is parabolic iff $(d - a)^2 + 4bc$ follows from the
quadratic formula.

(ii) $\gamma_1, \gamma_2$ are Möbius transformations precisely when $k, \ell > 0$. (Use ‘$ad - bc > 0$’ for this.)
To check when they are parabolic, the easiest way is to use the result in part (i). You
can use ‘parabolic iff $\tau(\gamma) = 4$’ - but YOU MUST NORMALISE (otherwise you
get the right answer but for completely the wrong, and invalid, reason). Alternatively,
you can just solve $\gamma_1(z_0) = z_0$ and determine when the resulting quadratic has just one
real solution - this is exactly the same as the argument in part (i), so in some sense it’s
wasted effort.

(iii) I think everybody who did it correctly checked that $\gamma_1, \gamma_2$ map $s_1$ to $s_2$ and $s_4$ to $s_3$ by
checking that $\gamma_1(1) = 1, \gamma_1(\infty) = 3/2$, and similarly for $\gamma_2$.
There are 3 parabolic cycles: 1, 2, and $\infty \rightarrow 3/2$ with parabolic cycle transformations
$\gamma_1, \gamma_2$ and $\gamma_2^{-1} \gamma_1$. 3
For the parabolic cycle condition to hold, you need all three parabolic cycle transformations to be parabolic (or the identity). For $\gamma_1, \gamma_2$ to be parabolic, we need $k = \ell = 1$. Then we need to check that $\gamma_2^{-1}\gamma_1$ is parabolic for these values of $k, \ell$. One cannot argue—as several did—that the composition of two parabolic transformations is parabolic (because this is not true). Instead, note that when $k = \ell = 1$, $\gamma_2^{-1}\gamma_1$ has matrix
\[
\begin{pmatrix}
-5 & 8 \\
-2 & 3 \\
\end{pmatrix}
\begin{pmatrix}
3 & -2 \\
2 & -1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 2 \\
0 & 1 \\
\end{pmatrix}
\]
which corresponds to $z \mapsto z + 2$, which is parabolic as it’s a translation.
Thus when $k = \ell = 1$, $\gamma_1, \gamma_2$ generate a Fuchsian group. As there are no relations (as there are no elliptic cycles) and two generators, the group is the free group on 2 generators.

C8  
(i) This is a standard definition from the course.
(ii) This is straightforward from the definitions and I think everyone got it right.
(iii) This is also straightforward: suppose $(a+i+b)/(c+i+d) = i$ and $ad - bc = 1$. Then $ai + b = di - c$ so that $a = d, b = -c$. Hence $a^2 + b^2 = 1$. As $0 \leq a \leq 1$, take $a = \cos \theta, b = \sin \theta$, and the first part follows. For the second part, note that the only integer solutions to $a^2 + b^2 = 1, ad - bc = 1$ are $a = \pm 1, b = 0$ and $a = 0, b = \pm 1$. This gives that the stabiliser of $i$ in $PSL(2, \mathbb{Z})$ is $\{z \mapsto z, z \mapsto -1/z\}$.
(iv) Take $g(z) = z + 1$ so that $g(i) = 1 + i$. Using parts (ii) and (iii) give the answer. (As an aside: there are lots of other choices for $g$ that map $i$ to $1 + i$; the answer you get is independent of this choice.)

C9  
(i) This is a standard definition from the course.
(ii) This is a proof from the notes.
(iii) Most of you got this right, and there are several ways of doing it.
    Let $b, d \in \mathbb{Z}, d \neq 0$. By the Euclidean algorithm, there exist $a, c \in \mathbb{Z}$ such that $ad - bc = 1$. Let $\gamma(z) = (az + b)/(cz + d) \in PSL(2, \mathbb{Z})$. Then $\gamma(0) = b/d$. Hence for all $x \in \mathbb{Q}$, there exists $\gamma \in PSL(2, \mathbb{Z})$ such that $\gamma(0) = x$. Hence the orbit of 0 contains $\mathbb{Q}$, which is not discrete.
(iv) Many of you got this wrong and claimed that it did act properly discontinuously.
    Just note that $\gamma_n(\infty) = \infty$ for all $n$, so $\text{Stab}_\Gamma(\infty)$ is infinite.

C10  
(i) This is a standard definition from the course.
(ii) This is a standard result from the course.
(iii) Note that (for any $z \in \mathbb{H}$) $\gamma_n(z) \to -1$ as $n \to \infty$ and $\gamma_n(z) \to 0$ as $n \to -\infty$. Hence $\Lambda(\Gamma) = \{-1, 0\}$. Surprisingly few of you correctly answered this.

Charles Walkden  
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