Feedback on Final Exam

My comments appear in italics

THE UNIVERSITY OF MANCHESTER

RANDOM MODELS

20 May 2015
14:00 – 16:00

Answer ALL questions in Section A and TWO questions in Section B. If all questions in Section B are attempted, then credit will be given for the best TWO answers.

Most questions are similar in style to examples and questions you have seen in Lectures and in example classes, as well as on past exam papers. Comparing to last year’s exam there are less proofs and more practical questions. I am happy that most of you made a good attempt to the questions, and quite a few of you achieved really high marks.

Electronic calculators may be used, provided that they cannot store text.
SECTION A

Answer **ALL** four questions

A1. [10 marks]

(i) Let $X$ be a discrete random variable with probability mass function
\[ P(X = 2n + 3) = 2^{-n}, \quad n = 1, 2, \ldots. \]

Find the probability generating function of $X$. \[ [4 \text{ marks}] \]

*Most students did well here. A common mistake is to sum up from $n = 0$ instead of from $n = 1.*

(ii) Let $Y$ be a discrete random variable with probability generating function
\[ G_Y(s) = \frac{1}{3}e^{s^3}(s + s^2 + s^3). \]

Find the probability mass function of $Y$. (You may use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$) \[ [6 \text{ marks}] \]

*Most students were able to develop $e^{s^3}$ into power series, but only some were able to write it as $\sum_{k=0}^{\infty} \frac{1}{3k!} (s^{3k+1} + s^{3k+2} + s^{3k+3})$ and deduce that $P(Y = 3k + 1) = P(Y = 3k + 2) = P(Y = 3k + 3) = \frac{1}{3k!}$ for $k = 0, 1, 2, \ldots$. A common mistake here is missing the term $k = 0, 1, 2, \ldots.*

A2. [10 marks]

Let $\{S_n, n \geq 0\}$ be a random walk starting from zero, i.e., $S_0 = 0$ and for $n \geq 1$, $S_n = \sum_{i=1}^{n} Y_i$, where $Y_1, Y_2, \ldots$ are independent with $P(Y_i = 1) = p, P(Y_i = -1) = q, p + q = 1$.

(i) Write down the probability mass function of the simple random walk, i.e., the probabilities $P(S_n = k)$ for $n \geq 1$ and $-n \leq k \leq n$. \[ [5 \text{ marks}] \]

*Generally done well. A common mistake is forgetting the case when $n + k$ is odd, $P(S_n = k) = 0.*

(ii) Compute the conditional probability:
\[ P(S_8 = 2, S_6 = 4 | S_4 = 2, S_2 = 0). \]

[5 marks]

*Generally done well. A few students applied the Markovian and homogenous properties in the wrong way.*

A3. [10 marks]
Suppose that $Y, N, N_1, N_2, \cdots$ are independent random variables taking values in non-negative integers, and $N_1, N_2, \cdots$ have the same distribution as $N$. Define the random sum $Z$ by

$$Z = N_1 + \cdots + N_Y = \begin{cases} N_1 + \cdots + N_r, & \text{if } Y = r \geq 1; \\ 0, & \text{if } Y = 0. \end{cases}$$

Suppose that $P(N = 0) = \frac{1}{3}$, $P(N = 1) = \frac{2}{3}$ and $P(Y = 0) = \frac{1}{4}$, $P(Y = 2) = \frac{3}{4}$.

(i) Find the probability generating function of $Z$. 

Generally done well. A few students made mistakes in calculations.

(ii) Calculate $E(Z)$ and $E(Z^2)$.

Most students were able to calculate $E(Z)$ using $E(Z) = G'_Z(1)$, but only a few students used the formula $E(Z^2) = G''_Z(1) + G'_Z(1)$ correctly.

A4. [10 marks]

Let $\{N(t), t \geq 0\}$ be a Poisson process of rate $\lambda$.

(i) Find the probability $P(N(t_1) = 1, N(t_2) = 2, N(t_3) = 3)$ for $0 \leq t_1 < t_2 < t_3$.

Generally done well. A few students make mistakes in calculations.

(ii) Find $P(N(s) = m|N(t) = n)$ for $s < t$ and $m < n$.

Quite a few students missed the condition $s < t$ and directly used homogeneity incorrectly. Note that $P(N(s) = m|N(t) = n) = P(N(s) - N(t) = m - n)$ holds only when $s > t$ and $m \geq n$. 
SECTION B

Answer TWO of the three questions

B5. [20 marks]
A drunk man keeps wandering on the integers \( \mathbb{Z} = \{ \cdots, -2, -1, 0, 1, 2, \cdots \} \). He starts from 0, and at each step, he moves one step forward (+1) with probability \( p \) and he moves one step backward (−1) with probability \( q = 1 - p \). Suppose that all movements are independent of each other.

(i) Conditioning on the event that the drunk man arrived at 19 at step 23, find the probability that the drunk man revisits 19 at step 31?

(ii) For which values of \( p \) is the probability that the drunk man ever visits 20 at least \( 4^{-10} \)?

(iii) If \( p = \frac{3}{4} \), find the probability that the drunk man visits 3 before visiting −5.

(iv) If \( p = \frac{2}{5} \), find the probability that the drunk man revisits 0 exactly 7 times?

(v) For \( a, b \in \mathbb{Z} \) let \( g_{a,b} \) denote the probability that the drunk man starting from \( a \) ever visits \( b \). Show that if \( a < b < c \), then \( g_{a,c} = g_{a,b} \cdot g_{b,c} \).

Generally done ok. Mistakes were made in calculations. A few students used the wrong formula. Common mistakes include obtaining the inequality \( p \geq 1/3 \) in the wrong direction in question (ii) and forgetting to mention “the drunk man must visit \( b \) from \( a \) to \( c \)” in question (v).

B6. [20 marks]
A zombie potion can turn you into a zombie for one day. During that day, you will be able to find \( N \) people to bite and turn them into zombies also for one day, where \( N \) is a random variable with probability mass function \( P(N = k) = p(1-p)^k \) for \( k = 0, 1, 2, \ldots \) and some \( p \in (0, 1) \). Suppose that all zombies have the same ability to find people to bite and their activities are independent of each other.

Suppose that you just drunk a zombie potion by accident. For \( n \geq 1 \) denote by \( Z_n \) the number of zombies of generation \( n \) descended from you.

(i) Find the probability generating function of \( Z_1 \).

(ii) Find the probability \( P(Z_2 = 0) \).

(iii) Find the expectation \( E(Z_3) \).

(iv) Find the extinction probability of the zombie population descended from you.
(v) If \( p = \frac{1}{2} \), show by induction that for \( n \geq 1 \),
\[
G_{Z_n}(s) = \frac{n - (n - 1)s}{n + 1 - ns}.
\]

[6 marks]

Questions (i) and (ii) were done well. In question (iii) a lot of students tried to find \( E(Z_3) \) by calculating \( G'_{Z_3}(1) \), the calculation is quite heavy and almost no one calculated it correctly. The right way to do it is to use the formula \( E(Z_3) = E(Z_1)^3 \). Questions (iv) and (v) are done ok.

B7. [20 marks]

Doctor Who is able to regenerate after suffering illness, mortal injury or old age. The process repairs all damage and rejuvenates his body, but as a side effect it changes his physical appearance and personality. Assume that Doctor Who first appears at time 0. For \( t \geq 0 \) denote by \( N(t) \) the number of regenerations Doctor Who has undergone by time \( t \). For \( k = 1, 2, \ldots \) denote by \( X_k \) the duration of Doctor Who’s \( k \)-th appearance. Then by definition
\[
N(t) = 0 \text{ for } t \in [0, X_1) \text{ and } N(t) = \max\{k \geq 1 : X_1 + \ldots + X_k \leq t\} \text{ for } t \in [X_1, \infty).
\]

For \( n \geq 0 \) let \( T_n \) be the time at which Doctor Who regenerates for the \( n \)-th time. Then by definition
\[
T_n = \inf\{t \geq 0 : N(t) = n\}.
\]

Suppose that \( X_1, X_2, \ldots \) are independent random variables, each exponential distributed with parameter \( \frac{1}{100} \).

(i) Find the conditional probability \( P(N(1200) = 11 \mid N(900) = 9) \). \([4 \text{ marks}]\)

(ii) Show that for \( t \geq 0 \) and \( k \geq 0 \), \( P(N(t) = k) = P(T_k \leq t) - P(T_{k+1} \leq t) \). \([4 \text{ marks}]\)

(iii) Show that for \( t \geq 0 \), \( T_{N(t)+1} - t \) has exponential distribution with parameter \( \frac{1}{100} \). \([4 \text{ marks}]\)

(iv) Find the expectation \( E(N(100)N(200)) \). \([4 \text{ marks}]\)

(v) Find the expected duration of the appearance of Doctor Who observed at time \( t = 1000 \). \([4 \text{ marks}]\)

Generally done well here. Common mistakes include forgetting to mention “\( N(t) \geq k \) if and only if \( T_k \leq t \)” in question (ii) and “\( T_{N(t)+1} - t > x \) if and only if no regeneration occurs in \( (t, t+x] \)” in question (iii). Some students made mistakes in calculating \( E(N(100)N(200)) \). Questions (i) and (v) are done well.

END OF EXAMINATION PAPER

5 of 5