Mistake in B8  As I’m sure you’re aware, there was a typographical mistake in one of the displayed equations in B8. A factor of 2 was missing and the integral in part (iii) should have read $\pi i/2\sqrt{2}$. The School has several processes by which exam papers are checked and all of them missed this; ultimately the responsibility is mine and I would like to apologise to the class for this oversight. When I marked the scripts I took careful note to see the extent to which the class might have been disadvantaged so that appropriate compensation could be applied. Of those (around 40% of the class) of you who answered B8, there was no pattern to suggest that the class as a whole had been significantly affected by this mistake.

Feedback on the exam  Most people answered B6 and B7. B8 was the next most popular question. B5 was least popular.

B5  Very few of you answered this question; there were also comparatively few complete answers. I’m not sure why - perhaps because it’s the one ‘least-like’ past exam paper questions, or perhaps because the complex logarithm is quite a difficult concept.

(i) This is largely bookwork.

The question says: ‘Let $z \in \mathbb{C} \setminus \{0\}$. Suppose that $\exp w = z$. Determine the real and imaginary parts of $w$.’ Some of you wrote: let $w = x + iy$, then the real part of $w$ is $x$ and the imaginary part of $w$ is $y$. The question is asking you to find the real and imaginary parts of $w$ in terms of $z$. To do this, write $w = x + iy$. Then $z = e^w = e^{x+iy} = e^x (\cos y + i \sin y)$. Then $|z| = e^x$ so that $x = \log |z|$ (this is now a real logarithm). We also have that $y$ is a value of the argument of $z$.

Hence $\log z = \log |z| + i \arg z$. The principal value of the logarithm is $\Log z = \log |z| + i \arg z$ where $\arg(z) \in (-\pi, \pi]$.

The principal logarithm is not continuous on $\mathbb{C}$ (perhaps I should have said $\mathbb{C} \setminus \{0\}$ as the logarithm isn’t even defined at 0) as the principal value of the argument is not continuous: it changes from near $-\pi$ to $\pi$ as one crosses the negative real axis. Some of you said that the principal logarithm is not defined on the negative real axis: yes it is (it corresponds to the case $\arg(z) = \pi$ above, for example $\Log(-2) = \log(2) + i\pi$), although we later restrict the principal logarithm to the cut-plane.

(ii) This is very similar to one of the exercises.

The question asks you to find two values of $z_1, z_2$ with a certain property. So you actually have to find examples of two points $z_1, z_2$. You won’t get anywhere by starting with $\Log \frac{z_1}{z_2} = \Log z_1 - \Log z_2$ and manipulating it.

There are lots of examples that work; to find one you need to have a good understanding of what the argument of a complex number is.

For example, take $z_1 = -i, z_2 = i$. Then $z_1/z_2 = -1$ which has modulus 1 and principal argument $\pi$. Hence $\Log(z_1/z_2) = \Log(-1) = \log |\{-1\}| + i\pi = i\pi$. However, $\Log z_1 = \log |\{-i\}| - i\pi/2 = -i\pi/2$ and $\Log z_2 = \log |i| + i\pi/2 = i\pi/2$. Hence $\Log z_1 - \Log z_2 = -i\pi \neq \Log z_1/z_2$.

(iii) Most of you got the definition of differentiability at $z_0$ correct, but you need to remember to take the limit. The definition is

$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, not $f'(z_0) = \frac{f(z) - f(z_0)}{z - z_0}$.
The proof that the derivative of Log $z$ on the cut-plane is $1/z$ is bookwork.

(iv) This was very poorly answered. By definition $i^{1+i} = \exp((1 + i) \log i)$. Now

$$\log i = \log |i| + i \left( \frac{\pi}{2} + 2n\pi \right) = i \left( \frac{\pi}{2} + 2n\pi \right), \quad n \in \mathbb{Z}$$

as $\arg(i) = \frac{\pi}{2} + 2n\pi$. The principal value of the logarithm occurs when $n = 0$. Substituting this in we obtain

$$i^{1+i} = ie^{-\left( \frac{\pi}{2} + 2n\pi \right)}, \quad n \in \mathbb{Z}$$

(using $e^{2n\pi i} = 1$ and $e^{i\pi/2} = i$), which are purely imaginary. The principal value is $ie^{-\pi/2}$.

B6 The vast majority of you attempted this question. There were some very good answers, but there were several very common misunderstandings and mistakes which I’ve detailed below.

(i) This is essentially bookwork: it’s asking you to reproduce the second half of the proof of the Cauchy-Riemann equations.

Some of you first proved that $f'(z_0) = \frac{\partial u}{\partial x}(z_0) + \frac{\partial v}{\partial x}(z_0)$ and then used the Cauchy-Riemann equations to deduce the claimed result. Credit was given for this but it was not the way the question was intended to be answered.

(ii) The following point was stressed several times in the lectures and in the support classes:

if a function is differentiable at a point then the Cauchy-Riemann equations hold at that point. The converse is not true: if the Cauchy-Riemann equations hold at a point then it is not necessarily true that the function is differentiable at that point.

However, if the partial derivatives if $u,v$ exist at $z_0$ and are continuous at $z_0$ and satisfy the Cauchy-Riemann equations then $f$ is differentiable at $z_0$.

There were many strange answers here, including several of you proving that the partial derivatives satisfy Laplace’s equation. This has no relevance to the question.

(iii) This is very similar to several exercises in the course and most people answered it correctly. Firstly you need to remember how to differentiate quotients using the quotient rule (it is not the case that $(f/g)' = f'/g'$). One obtains

\[
\frac{\partial u}{\partial x} = \frac{-x^2 + (y - 1)^2}{(x^2 + (y - 1)^2)^2} \\
\frac{\partial u}{\partial y} = \frac{-2x(y - 1)}{(x^2 + (y - 1)^2)^2} \\
\frac{\partial v}{\partial x} = \frac{2x(y - k)}{(x^2 + (y - 1)^2)^2} \\
\frac{\partial v}{\partial y} = \frac{-x^2 - (y - 1)^2 + 2(y - k)(y - 1)}{(x^2 + (y - 1)^2)^2}.
\]

For the Cauchy-Riemann equations to hold we need $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This implies $k = 1$.

When $k = 1$ the other Cauchy-Riemann equation $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ holds.

As the partial derivatives are continuous, part (ii) then implies that $f$ is differentiable.

(Many of you missed out this part. Remember: the Cauchy-Riemann equations holding on their own does not imply that the function is differentiable. Note that the question
explicitly encourages you not to make this mistake by asking you separately when the 
Cauchy-Riemann equations hold and when the function is differentiable.)

Another common mistake was to find values of $k$ that depended on $x$ and $y$; note that 
$k$ is a constant (the question says ‘value of $k$’).

(iv) Those of you who answered this question normally did well. Some common mistakes
were the following:

At the origin, one needs to calculate $\frac{\partial u}{\partial x}$ from first principles using the definition given
in the question; one cannot calculate $\frac{\partial u}{\partial x}$ by writing
$u(x,y) = \frac{(x^3 - y^3)}{(x^2 + y^2)}$, differentiating this, and then letting $x, y \to 0$. One obtains

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{h^3}{h} = \lim_{h \to 0} \frac{h^3}{h^3} = 1.$$ 

(And similarly for the other partial derivatives.) Note that $\lim_{h \to 0} \frac{h^3}{h^2}$ is undefined. 

For the final part, one obtains a formula for $\lim_{h \to 0} f(h+iab) - f(0)$ which depends on $a$. Hence $f$ cannot be differentiable at 0. This does not contradict (ii) as the partial 
derivatives are not continuous at 0.

B7 The vast majority of you attempted this question. Whilst there were a large number of very 
good answers, I was concerned by how many mistakes there were in understanding contour 
integration. In particular, there seems to be a wide-held belief that every function has an 
anti-derivative. This is not the case in complex analysis (indeed, this is the main point in 
complex analysis - if every function did have an anti-derivative then Cauchy’s Theorem, the 
Residue Theorem, etc, would all be very dull) - a point I stressed in the lectures many times.

(i) This is a standard definition from the course: $\int_C f = \int_a^b f(\gamma(t))\gamma'(t)\,dt$ if $\gamma : [a,b] \to \mathbb{C}$
is a parametrisation of $\gamma$.

(ii) To use the Estimation Lemma, you need (i) an upper bound on the integrand, (ii) the
length of the contour. The length of $C_4$ is $8\pi$ (use circumference = $2\pi r$ - some of you 
thought it was $\pi r^2$). To find an upper bound on the integrand, you want to make the 
numerator (in modulus) as large as possible and the denominator (in modulus) as small 
as possible. 

Suppose $z$ is a point on $C_4$. Then $|z| = 4$. 

By the triangle inequality, $|z + 1| \leq |z| + 1 = 4 = 1 = 5$. 

By the reverse triangle inequality, $|z^2 - 3z + 2| \geq |z^2| - |3z + 2|$. By the triangle 
inequality we have $|z^2| - 3|z| \leq |z^2| - 3|z| = |z^2| - 3|z| - 2 = 4^2 - 3 \times 4 - 2 = 2$. Note that as we are trying to find an upper bound for the 
integrand, we need to find a lower bound for the denominator; if you say something like 
$|z^2 - 3z + 2| \leq |z^2| + 3|z| + 2 = 30$ then, when you take the reciprocal, the inequalities 
go the wrong way.

Hence

$$\left| \frac{z + 1}{z^2 - 3z + 2} \right| \leq \frac{5}{2}.$$ 

By the Estimation Lemma,

$$\left| \int_{C_4} \frac{z + 1}{z^2 - 3z + 2} \, dz \right| \leq \frac{5}{2} \times 8\pi = 20\pi.$$ 

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There are other ways to bound the denominator. Note that $z^2 - 3z + 2 = (z - 2)(z - 1)$. Also note that if $z \in C_4$ then $|z - 2| \geq |z| - 2 = 4 - 2 = 2$ by the reverse triangle inequality. Similarly $|z - 1| \geq |z| - 1 = 4 - 1 = 3$. Hence $|z^2 - 3z + 2| = |z - 2| \times |z - 1| \geq 2 \times 3 = 6$.

(As an aside: you do need to bound the modulus of the integrand. You can only do inequalities with real numbers; you can’t do inequalities with complex numbers.)

(iii) The Fundamental Theorem of Contour Integration says that: Suppose $f : D \to \mathbb{C}$ is continuous and suppose that $f$ has an anti-derivative $F : D \to \mathbb{C}$. Suppose that $\gamma$ is a path from $z_0$ to $z_1$. Then

$$\int_{\gamma} f = F(z_1) - F(z_0).$$

Note that the hypotheses says ‘IF there is an anti-derivative’. Unlike in real analysis, most complex functions do not have anti-derivatives.

Saying that if there is an anti-derivative then the integral is independent of the choice of path also isn’t (quite) enough, as the FT of CI gives an explicit formula for the value of this integral.

(iv) Most of you answered this correctly.

The question asks you to evaluate $\int_{\gamma} f$ for any path from 0 to $i$. A common mistake was to just calculate $\int_{\gamma} f$ for one particular path, namely the path from 0 to $i$ that travels up the imaginary axis (i.e. the path parametrised by $\gamma(t) = it$, $0 \leq t \leq 1$). This isn’t what the question asks you.

Quite a few of you wrote down an antiderivative for $z^2 e^{iz}$ without showing any working.

Although I gave the benefit of the doubt, you really should show your working...

An anti-derivative for $z^2 e^{iz}$ can be found by integrating by parts (twice) to obtain $F(z) = -iz^2 e^{iz} + 2ze^{iz} + 2te^{iz}$. A common mistake was to lose an $i$ or a minus sign.

It’s then straightforward to calculate $F(i) - F(0) = 5ie^{-1} - 2i$.

B8 Despite the typographical error many of you got full, or close-to-full, marks on this question. I was pleased by how many of you got the correct answer for (iii), especially as it’s a slight generalisation of the method for computing trigonometric integrals that we discussed in class.

The worst-answered part of this question was the last part of (i): prove that if $z_0$ is a simple pole then $\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)f(z)$.

(i) $f$ has a singularity of $z_0$ if $f$ is not differentiable at $z_0$. I was pleased by how many of you got this right. When I’ve asked this question in previous years, many people have said that a singularity is when we divide by 0. Whilst this is often how singularities arise in the examples discussed in the course, this is not the definition.

If $f$ has a simple pole at $z_0$ then it has a Laurent series at $z_0$ of the form

$$\frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad (1)$$

with $b_1 \neq 0$, valid on some disc $0 < |z - z_0| < r$. Alternatively, the pole has order 1 (although answering this in the form (1) helps considerably with the next part of the question). Note that you can’t say that the principal part of the Laurent series contains only one term, as the term could be $b_2/(z - z_0)^2$ ($b_2 \neq 0, b_1 = 0$) which would give a pole of order 2.
If $f$ has a simple pole at $z_0$ then, from (1),

$$(z - z_0)f(z) = b_1 + \sum_{n=0}^{\infty} a_n(z - z_0)^{n+1} \to b_1 = \text{Res}(f, 0)$$

as $z \to z_0$.

(ii) Let $f(z) = 1/(z^2 + 6z + 1)$. This has singularities whenever the denominator vanishes. Now $z^2 + 6z + 1 = 0$ precisely when $z = -3 \pm 2\sqrt{2}$ and these zeros are simple. Hence $f$ has simple poles at $z = -3 \pm 2\sqrt{2}$. The pole at $-3 - 2\sqrt{2}$ lies outside the unit disc $C_1$, so we need only look at the pole at $-3 + 2\sqrt{2}$.

Using part (i), we have that

$$\text{Res}(f, -3 + 2\sqrt{2}) = \lim_{z \to -3+2\sqrt{2}} (z - (-3 + 2\sqrt{2})) \frac{1}{(z - (-3 + 2\sqrt{2}))(z - (-3 - 2\sqrt{2}))}$$

$$= \frac{1}{(-3 + 2\sqrt{2}) - (-3 - 2\sqrt{2})}$$

$$= \frac{1}{4\sqrt{2}}.$$

By the Residue Theorem,

$$\int_{C_1} f(z) \, dz = 2\pi i \times \frac{1}{4\sqrt{2}} = \frac{\pi i}{2\sqrt{2}}.$$

Several of you instead calculated $\int_{-\infty}^{\infty} \frac{1}{x^2+6x+1} \, dx$ using the ‘D-shaped contour’ argument. Whilst some of the mathematics is the same, this is not what the question asked you to do. The limits in the integral do matter!

(iii) This is a generalisation of the argument used in the course to calculate trigonometric integrals. Let $z = e^{2it}$ and note that as $t$ varies from 0 to $\pi$ then $z$ varies around the unit circle $C_1$, going around once anti-clockwise.

Note that $dz = 2ie^{2it} \, dt = 2iz \, dt$ so that $dt = \frac{dz}{2iz}$. Also

$$\cos^2 t = \left(\frac{e^{it} + e^{-it}}{2}\right)^2 = \frac{e^{2it} + 2 + e^{-2it}}{4} = \frac{z + 2 + z^{-1}}{4}.$$

The result now follows from carefully substituting in and some (easy) algebraic manipulation.

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