A Questions:

A1. (a) A sequence \((a_n)_{n \in \mathbb{N}}\) converges to a limit \(\ell\) if for all \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that \(|a_n - \ell| < \epsilon\) for all \(n \geq N\).

(b) \(a_n \to \infty\) as \(n \to \infty\) if for all real numbers \(K\) there exists \(N \in \mathbb{N}\) such that \(a_n > K\) for all \(n \geq N\).  

4 marks

A2. Note that
\[
\left| \frac{n^3}{3n-1} - \frac{1}{3} \right| = \left| \frac{3n^3 - 3n + 1}{3(3n-1)} \right| = \left| \frac{1}{9n-3} \right|.
\]
Hence
\[
\left| \frac{n^3}{3n-1} - \frac{1}{3} \right| < \epsilon \iff 9n - 3 > \frac{1}{\epsilon} \iff n > \frac{1}{9} \left( \frac{1}{\epsilon} + 3 \right).
\]
Thus one should take \(N = 1 + \left\lceil \frac{1}{9} (\frac{1}{\epsilon} + 3) \right\rceil\).  

6 marks

A3. Dividing top and bottom by the dominant term \(n^3\) and applying the Algebra of Limits Theorem gives
\[
\frac{(n+1)^3}{2n^3 + \left(\frac{1}{2}\right)^n} = \frac{(1 + \frac{1}{n})^3}{2 + \frac{1}{n^3\left(\frac{1}{2}\right)^n}} \to \frac{1}{2} + 0 = \frac{1}{2} \quad \text{as } n \to \infty.
\]

5 marks

A4. For \(k = 0\) the result is obvious, so suppose by induction that \(\lim_{n \to \infty} \frac{\ln(n)^k}{n} = 0\) for some \(k \geq 0\).

Since both \(f(n) = n\) and \(g(n) = \ln(n)^{k+1}\) are positive and tend to infinity as \(n \to \infty\), we may apply L’Hôpital’s Rule to get
\[
\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{g'(n)}{f'(n)} = \frac{(k+1) \frac{1}{n} \ln(n)^k}{1} = (k+1) \frac{\ln(n)^k}{n}.
\]
By induction this final term tends to zero and hence so does \(\frac{\ln(n)^{k+1}}{n}\). Thus by induction \(\lim_{n \to \infty} \frac{\ln(n)^m}{n} \to 0\) for all \(m \geq 0\).  

6 marks
A5. (a) Using the Integral test; Certainly \( f(x) = (x(\ln(x))^2)^{-1} \) is continuous, decreasing and positive, so the test applies. 

\[
\int_2^\infty \frac{1}{x(\ln(x))^2} \, dx = -\ln(x)^{-1}|_2^\infty = \ln(2)^{-1} < \infty.
\]
Hence the series converges.

(b) As \( \lim_{n \to \infty} \frac{n!}{e^n} = \infty \neq 0 \) the series diverges by the \( n \)-th term test.

(c) \( \frac{n^2}{n^3 + n} \leq \frac{n}{n^3} = n^{-2} \). Since \( \sum n^{-2} \) converges, so does \( \sum \frac{n^2}{n^3 + n} \) by the comparison test.

4 marks each

A6. (a) Write \( \frac{1}{4n^2 - 1} = \frac{A}{2n + 1} + \frac{B}{2n - 1} = \frac{(2n - 1)A + (2n + 1)B}{4n - 1} \).

From this we get \( A + B = 0 \) and \( B - A = 1 \); thus \( B = \frac{1}{2} \) and \( A = -\frac{1}{2} \). Thus

\[
\sum_{n=1}^t \frac{1}{4n^2 - 1} = \sum_{n=1}^t \frac{1}{2} \left( \frac{1}{2n - 1} - \frac{1}{2n + 1} \right) = \frac{1}{2} \left( \frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2t - 1} - \frac{1}{2t + 1} \right).
\]

The intermediate terms cancel and so

\[
\sum_{n=1}^\infty \frac{1}{4n^2 - 1} = \frac{1}{2} \left( 1 - \frac{1}{2t + 1} \right). \quad \text{Hence} \quad \sum_{n=1}^\infty \frac{1}{4n^2 - 1} = \frac{1}{2}.
\]

7 marks

Comments on Students’ solutions.

A Questions. These were mostly done fairly well. However, some comments are as follows.

A1. Definitions must be precise!
A2. There is no problem in giving less efficient bounds, so there are many possible solutions to this question.
A4. Note that \( \ln(x)^n \) means \( (\ln(x))^n \) not \( \ln(x^n) \). Students who did answer the problem for \( \ln(x^n) \) were able to get a substantial number of marks, but many failed to check whether the hypotheses of L’Hôpital’s theorem were satisfied. They are not when \( n = 0 \).
A5. One of the more common errors was to claim that the \( n \)-th term test says that if \( a_n \to 0 \) as \( n \to \infty \), then \( \sum a_n \) converges. This is not true—it says the converse: if \( \sum a_n \) converges then \( a_n \to 0 \) as \( n \to \infty \). A typical example is \( \sum \frac{1}{n} \).
B7. (i) **Proof:** Since \((b_n)\) is bounded, there exists \(B > 0\) such that \(|b_n| \leq B\) for all \(n\). Also, for any \(\eta > 0\) there exists \(N \in \mathbb{N}\) such that \(|a_n| < \eta\) for all \(n \geq N\).

So, if \(\epsilon > 0\) is given, take \(\eta = \frac{\epsilon}{B}\). Then with \(N\) as above we get

\[ |a_n b_n| = |a_n| \cdot |b_n| < \eta B = \epsilon, \]

as required.  

(ii) **No** For example take \(a_n = 1\) for all \(n\) and \(b_n = (-1)^n\). Then clearly \((a_n)\) is convergent yet \(a_n b_n = (-1)^n\) does not converge.  

(iii) As \(\sqrt{n^2 + n} \geq n\) it suffices to prove that \(\sqrt{n^2 + n} < n + 1\) or, equivalently, that \((n^2 + n) < (n + 1)^2 = n^2 + 2n + 1\). This is obvious.  

(iv) One subsequence is obtained for \(k_n = n^2\) in which case \(a_{k_n} = [\sqrt{n^2}] - \sqrt{n^2} = n - n^2 = 0\), with limit 0.

By part (iii) we should also try \(k_n = n^2 + n\). Here

\[ a_{k_n} = [\sqrt{n^2 + n}] - \sqrt{n^2 + n} = n - \sqrt{n^2 + n}. \]

Here we apply the standard technique:

\[ n - \sqrt{n^2 + n} = \frac{(n - \sqrt{n^2 + n})(n + \sqrt{n^2 + n})}{n + \sqrt{n^2 + n}} = \frac{n^2 - (n^2 + n)}{n + \sqrt{n^2 + n}} = \frac{-n}{n + \sqrt{n^2 + n}}. \]

This equals \(\frac{-1}{1 + \sqrt{1 + \frac{1}{n}}}\) which tends to \(-\frac{1}{2}\) as \(n \to \infty\).

Since we have two subsequences with different limits, the original sequence cannot converge.  

8 Marks

**Comments on Students’ solutions.** There were various errors in the solutions but no especially common ones
B8. (i) We first prove by induction that \(a_n < 1\) for all \(n \in \mathbb{N}\). Certainly \(a_1 < 1\), so suppose by induction that that \(a_n < 1\). Then \(3 - 2a_n > 0\) and so

\[
a_{n+1} = \frac{2 - a_n^2}{3 - 2a_n} < 1 \iff 2 - a_n^2 < 3 - 2a_n \quad \text{by the above comment}
\]

\[
\iff 0 < a_n^2 - 2a_n + 1 \iff 0 < (a_n - 1)^2 \quad \text{by collecting terms.}
\]

As \(a_n < 1\) (or as \(a_n \neq 1\)), this last assertion is true. Therefore, going backwards through the equivalences, we see that \(a_{n+1} < 1\). Hence the inductive statement is true for all \(n \geq 1\).

6 Marks

(ii) Next, as \(3 - 2a_n > 0\) for all \(n\), we get

\[
a_{n+1} = \frac{2 - a_n^2}{3 - 2a_n} \geq a_n \iff 2 - a_n^2 \geq 3a_n - 2a_n^2
\]

\[
\iff a_n^2 - 3a_n + 2 \geq 0 \iff (a_n - 1)(a_n - 2) \geq 0
\]

\[
\iff (1 - a_n)(2 - a_n) \geq 0
\]

Again this last line is true as \(a_n < 1\). Thus, \(a_{n+1} \geq a_n\) for all \(n \geq 1\).

6 Marks

(iii) By (i) and (ii) \((a_n)\) is increasing and bounded above. Thus, by the Monotone Convergence Theorem \(\lim_{n \to \infty} a_n\) exists; say \(\lim_{n \to \infty} a_n = \ell\).

Now, the Algebra of Limits Theorem says that the sequence \(b_n = \frac{2 - a_n^2}{3 - 2a_n}\) also has a limit which is \(\lim_{n \to \infty} b_n = \frac{2 - \ell^2}{3 - 2\ell}\). However, as \(b_n = (a_{n+1})\), a Lemma from the notes says that \(\lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n = \ell\). In other words, \(\ell = \frac{2 - \ell^2}{3 - 2\ell}\).

Solving we get \(2 - \ell^2 = 3\ell - 2\ell^2\) and hence \(2 - 3\ell + \ell^2 = 0\). Equivalently \(\ell = 2\) or \(\ell = 1\).

Finally, as \(a_n \leq 1\) for all \(n\) a Lemma from the notes shows that \(\ell \leq 1\) and hence \(\ell = 1\).

8 Marks

Comments on Students’ solutions.

Parts (i,ii) As I mentioned in class it is very important to write out these sorts of proofs carefully. In particular, if you replace \(\iff\) by \(\Rightarrow\) in the arguments above, then you really cannot draw any conclusions. The point is that you can prove anything you want starting with a false statement! Similarly do be careful that you do not use the conclusion of the question to prove something.

Part (iii) Here, it is important to mention the Monotone Convergence Theorem if you are using it. Also you must first argue that the sequence is convergent, since otherwise you cannot use the Algebra of Limits Theorem.
B9. (i) If the real number $R$ has the property that the series $S = \sum_{n=1}^{\infty} a_n x^n$ converges for all $x$ with $|x| < R$, and diverges for all $x$ with $|x| > R$, then $R$ is called the \textit{radius of convergence} of $S$. We say that $R = 0$ (respectively $R = \infty$) if $S$ converges only for $x = 0$ (respectively converges for all $x \in \mathbb{R}$).

The interval of convergence is the set of numbers $\{x : S = \sum_{n=1}^{\infty} a_n x^n \text{ converges}\}$.

4 Marks

(ii) We use the (modified) Ratio test: For the first series, the ratio $\frac{|a_{n+1}|}{|a_n|}$ is

\[
\frac{(n+1)!}{\sqrt{(2n+2)!}} |x|^{n+1} \cdot \frac{\sqrt{(2n)!}}{n!} |x|^{-n} = \frac{n+1}{\sqrt{(2n+2)(2n+1)}} \frac{|x|}{\sqrt{2 \frac{2}{n}} (2 + \frac{1}{n})}
\]

As $n \to \infty$ the Algebra of Limits says that this tends to $\frac{1 + \frac{1}{n}}{\sqrt{2 \frac{2}{n}} (2 + \frac{1}{n})} = \frac{1}{\sqrt{4}} |x| = \frac{|x|}{2}$. Hence the radius of convergence is 2.

5 Marks

(iii) For the second series the ratio of absolute values is

\[
\frac{\ln(n+1)}{(n+1)} |x|^{n+1} \cdot \frac{n}{\ln(n)} |x|^n = \frac{\ln(n+1)}{\ln(n)} \frac{n}{(n+1)} |x|.
\]

Now, clearly $\lim_{n \to \infty} \frac{n}{(n+1)} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 1$.

Similarly by L'Hôpital’s rule (or the hint) $\lim_{n \to \infty} \frac{\ln(n+1)}{\ln(n)} = \lim_{n \to \infty} \frac{(1+n)^{-1}}{n^{-1}} = 1$, again. Hence by the Algebra of Limits, the limit of the display is $1$. $|x| = |x|$ and so the radius of convergence is 1.

5 Marks

Now when $x = +1$ the series becomes $\sum (-1)^n \frac{\ln(n)}{n}$. By Question A4, $\ln(n) \to 0$ as $n \to \infty$ and these terms are certainly positive. So the Alternating Series Test says the series converges when $x = +1$.

2 Marks

When $x = -1$ the series becomes $\sum \frac{\ln(n)}{n}$. In this case, we use the integral test, which does apply since the terms clearly are decreasing and (for example Question A4) tend to 0. Here

\[
\int_{x=1}^{\infty} \frac{\ln(x)}{x} \, dx = \frac{1}{2} (\ln(x))^2 \bigg|_{x=1}^{\infty} = \infty.
\]

Hence the series diverges for $x = -1$. (In this case you could also use the comparison test by comparing the given series with $\sum \frac{1}{n}$, but you do need to say that $\ln(n) > 1$ for $n > 2$.)

3 Marks

Finally, this means that the interval of convergence is $(-1, 1]$.

1 Marks

Comments on Students’ solutions.

(i) Again, definitions need to be written carefully and precisely.

(ii) Don’t forget the absolute value signs!

(iii) Many people failed to test $x = \pm 1$ for convergence, or used the wrong test. Remember that when the signs of the $a_n$ alternate in a series $\sum a_n$, then the Alternating Series Test is always a good bet. But you must check that $|a_n| \to 0$ as $n \to \infty$.  