Tutorial on Constructive Set Theory, Mathlogaps workshop, Manchester, July, 2008

Peter Aczel

June 3, 2008

Constructive Set Theory (CST) was initiated by John Myhill by introducing in [9] an extensional set-theoretical formal system to represent the brand of constructive mathematics introduced by Errett Bishop in [5], updated in [6]. A closely related formal system, CZF, was introduced in [1], where it was given an interpretation in Martin-Löf's Intuitionistic Type Theory, [8]. In the papers [2, 3], extensions of CZF obtained by adding various axioms such as choice principles and the regular extension axiom, REA, were also shown to be interpreted. A generalisation of the interpretation may be found in [7].

Today CZF and CZF+REA are two key formal systems for CST. The formal system CZF is a proof-theoretically weak subsystem of ZF that uses intuitionistic logic, but has the same theorems as ZF when classical logic is used.

The aim of my first two tutorial lectures will be to introduce CST as a setting in which constructive mathematics can be developed and presented in essentially the same extensional style as classical mathematics is usually presented while only using principles that are generally accepted as constructively correct. To do constructive mathematics in this style it is only necessary to become sensitive to the need to reason only using intuitionistic logic and the 'predicative' set existence principles that are acceptable in CST. Although countable and dependent choices are usually taken to be constructively acceptable I will avoid their use as much as possible so as to be compatible with topos mathematics. Topos mathematics also only uses intuitionistic logic, but is thoroughly impredicative. A key result of CZF is the categorical axiomatisation of the Dedekind real numbers, with the construction of the reals done using Dedekind cuts; the more standard construction in constructive mathematics via Cauchy sequences only being equivalent when countable choice is assumed.

Also in the first two lectures I will give an informal description of the type theoretic interpretation of CST in Martin-Löf Type Theory that has been developed in [1, 2, 3, 7]. This is based on the constructive iterative, combinatorial notion of set that is intended to explain what CST is about. This can be used to justify the axiom system CZF as well as additional axioms such as REA.

In my remaining two lectures I will focus on a selection of more advanced topics of CST such as inductive and coinductive definitions and point-set and point-free topology. These topics and other topics concerning CST are treated in [4].

References

[1] P. Aczel: *The type theoretic interpretation of constructive set theory.* In: MacIntyre, A. and Pacholski, L. and Paris, J, editor, *Logic Colloquium '77* (North Holland, Amsterdam 1978) 55–66.

[2] P. Aczel: *The type theoretic interpretation of constructive set theory: Choice principles.* In: A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium* (North Holland, Amsterdam 1982) 1–40.

[3] P. Aczel: *The type theoretic interpretation of constructive set theory: Inductive definitions.* In: R.B. et al. Marcus, editor, *Logic, Methodology and Philosophy of Science VII* (North Holland, Amsterdam 1986) 17–49.

[4] P. Aczel, M. Rathjen: *Notes on constructive set theory*, Technical Report 40, Institut Mittag-Leffler (The Royal Swedish Academy of Sciences, 2001). *http://www.ml.kva.se/preprints/archive2000-2001.php*

[5] E. Bishop: *Foundations of Constructive Analysis.* McGraw-Hill, New york, 1967.

[6] E. Bishop and D. Bridges: *Constructive Analysis*. Springer, 1985.

[7] N. Gambino and P. Aczel. *The generalised type-theoretic interpretation of constructive set theory*. Journal of Symbolic Logic, 47(1):67-103,2006.

[8] P. Martin-Löf. *Intuitionistic Type Theory - Notes by G. Sambin of a series of lectures given in Padua, June 1980*. Bibliopolis, 1984.

[9] J. Myhill. *Constructive Set Theory.* Journal of Symbolic Logic, 40(3):347-382,1975.